Automatic Control and Biosystem

Adaptive Observers for Wastewater Treatment

Sergej Čelikovský

ÚTIA AV ČR, v.v.i.

FEKT VUT v Brně

4. prosince 2015

Contents

Static adaptive observers

Linear case

$$\dot{x} = Ax + bu + p\delta, \quad y = cx, \; x, b, c^\top, p \in R^n, y, u, \delta \in R.$$

State x, input u, output y, unknown constant perturbation p.
(A, c) detectable, i.e. $\exists l \in R$ such that $A + lc$ is Hurwitz.
Static adaptive observer:

$$\dot{\hat{x}} = [A + lc]\hat{x} + lc\hat{x} - ly + bu + p\phi, \; \dot{\phi} = y - c\hat{x},$$

ED: $\dot{e} = [A + lc]e + p(\phi - \delta), \quad e := \hat{x} - x, \quad \epsilon := \phi - \delta, \quad \dot{\epsilon} = \dot{\phi}.$

Assumption: $[c, A + lc, p]$ is strictly passive, i.e. $\exists P, Q > 0$:

$$[A + lc]^\top P + P[A + lc] < -Q, \quad Pp = c^\top, \quad P = P^\top.$$

ED strictly passive from “input” $\tilde{u} = \phi - \delta$ to “output” $\tilde{y} = ce$
with the **storage function** $W = (1/2)e^\top Pe$: $\dot{W} < \tilde{y}\tilde{u}, \quad \forall e \neq 0,$

$$\dot{W} = (1/2)e^\top[[A+lc]^\top P+P[A+lc]]e+e^\top Pp(\phi-\delta) < -(1/2)e^\top Qe+$$

$$e^\top Pp(\phi - \delta) < e^\top Pp(\phi - \delta) < e^\top c^\top(\phi - \delta) = ce(\phi - \delta) = \tilde{y}\tilde{u}$$

Static adaptive observers
Linear case

Use LF candidate $V := \epsilon^2/2 + (1/2)e^\top Pe$ to have:

$$\dot{V} = (1/2)e^\top[[A+lc]^\top P+P[A+lc]]e+e^\top Pp\epsilon-\dot{\epsilon}\epsilon < -(1/2)e^\top Qe+ [e^\top Pp-\dot{\epsilon}]\epsilon = -(1/2)e^\top Qe+[e^\top c^\top -ce]\epsilon = -(1/2)e^\top Qe, \ \forall e \neq 0.$$

By LaSalle principle, trajectory of $[e, \epsilon]^\top(t)$ converges to the largest invariant set inside the set $\dot{V} = 0$ which equals to $[0, \epsilon]^\top$, i.e. $e \to 0$ as $t \to \infty$. Next, dynamics on $\dot{V} = 0$ is

$$\dot{e} = p\epsilon, \quad \dot{\epsilon} = 0, \quad \text{i.e. } \epsilon(0) \neq 0 \ \Rightarrow e(t) \neq 0 \ \forall t \in (0, t^*), t^* > 0,$$

if $p \neq 0$. So, the only forward invariant subset of $\dot{V} = 0$ is the origin and therefore $\phi(t) \to \delta$ as $t \to \infty$.

Adaptive observer with unknown parameter identification.

Proposition: If $[c, A + lc, p]$ is strictly passive then adaptive observer with identification of unknown parameter is possible.

Static adaptive observers

Linear case

Necessary and sufficient condition for **“Assumption”** to hold with some fixed gains l is the famous **strictly positive real condition (SPR)**. Namely, roughly saying, the well-known KYP (Kalman Yacubovich Popov) Lemma states that those P, Q exist iff

$$\mathbf{Re}\ \left\{c[\omega \mathbf{i} I - A - lc]^{-1} p\right\} > 0\ \forall \omega \in R \quad (\Rightarrow rel.\ degree = 1).$$

Good for analysis, but how to design gains l? E.g., Marino-Tomei: assume (A, c) is in observer form, $p_1 s^{n-1} + \ldots + p_{n-1} s + p_n$ is Hurwitz and $p_1 > 0$. Take

$$l = \frac{1}{p_1}(Ap + \lambda p),\ \ \lambda > 0 \quad \Rightarrow$$

$$p_1(s^n + l_1 s^{n-1} + \ldots + l_n) = (s+\lambda)(p_1 s^{n-1} + \ldots + p_n) \quad \Rightarrow$$

$$c[sI - A - lc]^{-1} p = \frac{p_1}{s+\lambda} \quad \Rightarrow \quad \mathbf{SPR}$$

Static adaptive observers
Linear case

Marino-Tomei condition sufficient only:

1. KYP Lemma requires special form of Q.
2. SPR conditions are sufficient only.

Example: Two dimensional system in the observer form.

$$c = [1, 0], \quad A = \begin{bmatrix} l_1 & 1 \\ l_2 & 0 \end{bmatrix}, \quad p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

$$\mathbf{Re}\ \left\{c[(\omega \mathbf{i})I - A - lc]^{-1}p\right\} = \mathbf{Re}\ \left\{[1, 0] \begin{bmatrix} (\omega \mathbf{i}) - l_1 & -1 \\ -l_2 & (\omega \mathbf{i}) \end{bmatrix}^{-1} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}\right\}$$

$$\mathbf{Re}\ \left\{\frac{[1, 0] \begin{bmatrix} (\omega \mathbf{i}) & 1 \\ l_2 & (\omega \mathbf{i}) - l_1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}}{-\omega^2 - l_1 \omega \mathbf{i} - l_2}\right\} = \mathbf{Re}\ \left\{\frac{p_1 \omega \mathbf{i} + p_2}{-\omega^2 - l_1 \omega \mathbf{i} - l_2}\right\} =$$

Static adaptive observers
Linear case

$$\textbf{Re}\left\{\frac{(p_1\omega\mathbf{i}+p_2)(-\omega^2+l_1\omega\mathbf{i}-l_2)}{(\omega^2+l_2)^2+l_1^2\omega^2}\right\}=\frac{-l_2p_2+\omega^2(-p_2-l_1p_1)}{(\omega^2+l_2)^2+l_1^2\omega^2}.$$

So the corresponding transfer function is SPR if and only if (recall that $l_{1,2}<0$ as these are observer gains)

$$-l_2p_2>0\wedge -p_2-l_1p_1>0 \Leftrightarrow p_2>0\wedge p_1>-p_2/l_1>0 \Leftrightarrow |l_1|>p_2/p_1,$$

Summarizing: if $n=2$ and A,c is in observer form, SPR holds ONLY IF $p_{1,2}>0$. Moreover, for any given $p_{1,2}>0$ SPR can be satisfied by taking observer gains $l_{1,2}<0$ such that $|l_1|>p_2/p_1, l_2\in R$.
That is, Marino-Tomei condition is necessary and sufficient for two dimensional system.

Static adaptive observers

Measurable nonlinearities

Consider the system (see again Marino-Tomei):

$$\dot{x} = Ax + f(y,u,t) + p\,\beta(y,u,t)\delta, \quad y = cx, \quad c = [1,0,\dots,0],$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \delta = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_q \end{bmatrix} \in R^q, p = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \in R^n.$$

Here β is a q-row vector function , i.e. $\beta(y,u,t)\delta$ is a scalar quantity.

Static adaptive observers

Measurable nonlinearities

Adaptive observer:

$$\dot{\hat{x}} = [A + lc]\hat{x} + lce + f(y,u,t) + p\,\beta(y,u,t)\phi, \quad e := \hat{x} - x,$$

$$\dot{\phi} = -\Gamma\beta(y,u,t)ce, \quad \phi \in R^q, \quad \Gamma = \Gamma^\top > 0.$$

$$\dot{e} = [A + lc]e + p\beta(y,u,t)(\phi - \delta)$$

$$\epsilon := \phi - \delta, \quad \dot{\epsilon} = \dot{\phi}, \quad \Rightarrow \quad \dot{\epsilon} = -\Gamma\beta(y,u,t)ce.$$

Provided those P, Q as in linear case exist, Lyapunov function $(1/2)\epsilon^\top\Gamma\epsilon + (1/2)e^\top Pe$ proves that $e(t) \to 0$ as $t \to \infty$. LaSalle principle can not be applied for autonomous systems. Indeed:

$$\dot{e} = p\beta(y,u,t)\epsilon, \quad \epsilon \in R^q, \quad \text{for} \quad e = 0.$$

Proposition: Assume $\beta(y(t),u(t),t)$ is persistently exciting (PE), i.e. $\exists t_0, k_0, T > 0$ s.t. $\int_t^{t+T} \beta^\top(\tau)\beta(\tau)\mathrm{d}\tau \geq k_0 I > 0, \ \forall t \geq t_0$. Then $\lim_{t\to\infty} \epsilon(t) = 0$.

Proof is not easy and checking PE assumption is difficult.

Dynamic adaptive observers
Linear case

Consider SISO system

$$\dot{x} = Ax + bu + p\delta, \quad y = cx, \quad x, b, p \in R^n,$$

y, u, δ are scalar output, input and constant parameter perturbation. A, c detectable: $\exists l \in R$, s.t. $A + lc$ is Hurwitz.
Adaptive observer:

$$\dot{\hat{x}} = [A + lc]\hat{x} + lce + bu + p\phi + \gamma(t)\dot{\phi}, \quad e := \hat{x} - x,$$

$$\dot{\phi} = -\gamma^\top(t)c^\top ce$$

$$\dot{\gamma}(t) = [A + lc]\gamma(t) + p.$$

$\gamma(t) \in R^n$ is a column vector solution, called *adaptive gains*.
In total, $2n + 1$ equations: n observer equations, one parameter adaptation equation, n adaptive gains adjustement equations.

Dynamic adaptive observers
Linear case

Error dynamics:

$$\dot{e} = [A + lc]e + p(\phi - \delta) + \gamma(t)\dot{\phi}$$

$$\epsilon := \phi - \delta, \quad \dot{\epsilon} = \dot{\phi}.$$

Introduce the *combined error* $\bar{e}$:

$$\bar{e} = e - \gamma(t)\epsilon, \quad \dot{\bar{e}} = \dot{e} - \dot{\gamma}(t)\epsilon - \gamma\dot{\epsilon} =$$

$$[A+lc]e+p(\phi-\delta)-\gamma(t)\dot{\phi}-\dot{\gamma}(t)\epsilon-\gamma\dot{\epsilon} = [A+lc]e+p(\phi-\delta)-\dot{\gamma}(t)\epsilon$$

$$\dot{\bar{e}} = [A+lc]e+p(\phi-\delta)-[[A+lc]\gamma(t)+p]\epsilon = [A+lc]e-[A+lc]\gamma(t)\epsilon.$$

$$\dot{\bar{e}} = [A + lc]\bar{e}.$$

Therefore, $\bar{e} \to 0$ exponentially as $t \to \infty$.

Dynamic adaptive observers
Linear case

Adaptation equation can be rewritten as

$$\dot{\epsilon} = -\gamma^\top(t)c^\top c\gamma(t)\epsilon - \gamma^\top(t)c^\top c\bar{e}$$

Assumption: $c\gamma(t)$ is PE (persistently exciting), i.e.:

$$\exists\alpha > 0,\ T > 0,\ t_0 \geq 0: \quad \alpha \leq \int_t^{t+T} \gamma^\top(s)c^\top c\gamma(s)\mathrm{d}s, \quad \forall t \geq t_0.$$

If PE holds, then $\epsilon(t) \to 0$ as $t \to \infty$ and therefore $e = \bar{e} + \gamma(t)\epsilon$ goes to zero as t goes to infinity.

Remark: Only adaptation with parameter identification is possible!

Multi-output case: more adaptation equations.
Multi-perturbations case: analogous, $p \to$ matrix P, $\gamma \to$ matrix Γ, PE more complicated. Adding suitable time varying matrix on the adaptation eq. rhs may help to satisfy PE property.

Dynamic adaptive observers
Linear case

Scalar output and scalar perturbation case: only helps to improve the convergence speed only provider PE holds.
Scalar output and scalar perturbation case: PE property has nice interpretation.

$$\gamma^\top(t)c^\top c\gamma(t) = (c\gamma(t))^2$$
$$\dot\gamma(t) = [A + lc]\gamma(t) + p.$$

As $A + lc$ is Hurwitz, it holds $\lim_{t\to\infty} \gamma(t) = -[A + lc]^{-1}p\ \forall\gamma(0)$.
Therefore, PE is equivalent to the property that

$$c[A + lc]^{-1}p \neq 0.$$

Remark: Necessary and sufficient for the existence of the dynamic adaptive observer. It means that $c[sI - A - lc]^{-1}p$ is nonzero for $s = 0$ (no zero at ∞). Much weaker that SPR requiring $\mathbf{Re}\left\{c[sI - A - lc]^{-1}p\right\} > 0$ on the whole imaginary axis.

Dynamic adaptive observers
Linear case

Example:

$$\dot{x}_1 = x_2 + p_1\delta, \quad \dot{x}_2 = p_2\delta, \quad y = x_1, \qquad \rightarrow \qquad \frac{p_1 s + p_2}{s^2 - l_1 s - l_2}$$

Static AO exists **if** $p_2 > 0, p_1 > 0$. Dynamic AO **iff** $p_2 \neq 0$.
Example:

$$\dot{x}_1 = x_2 + \delta, \quad \dot{x}_2 = 0, \quad y = x_1.$$

System can not be in any way observed, as $(d/dt)(x_2 + \delta) = 0$, *i.e.*, impossible to know δ and x_2 separately. Indeed: $c[A + lc]^{-1}p = 0$ regardless the selection of observer gains l.
Example:

$$\dot{x}_1 = x_2 + \delta, \quad \dot{x}_2 = \delta, \quad y = x_1.$$

AO possible as $c[A + lc]^{-1}p \neq 0$ for any selection $l \neq 0$.

Q. Zhang, "Adaptive observer for multiple-input-multiple-output (mimo) linear time-varying systems," IEEE Trans. Autom. Control, 47 (2002), no. 3, pp. 525–529.

Dynamic adaptive observers

Measurable nonlinearities

Nonlinear system

$$\dot{x} = Ax + bu + f(y,u,t) + p\delta, \quad y = cx.$$

Adaptive observer:

$$\dot{\hat{x}} = [A + lc]\hat{x} + lce + bu + f(y,u,t) + p\phi + \gamma(t)\dot{\phi}, \quad e := \hat{x} - x,$$

$$\dot{\phi} = -\gamma^\top(t)c^\top ce$$

$$\dot{\gamma}(t) = [A + lc]\gamma(t) + p.$$

The rest is the same.
Perhaps, the most general case where PE can be checked through some clear test.
Again, MIMO and multi-perturbation cases are analogous.

Dynamic adaptive observers

Measurable nonlinearities in perturbation term as well

System

$$\dot{x} = A(t,y,u)x + b(y,t,u)u + f(y,u,t) + p(t,y,u)\delta, \quad y = cx.$$

If $\exists l$ s.t. $\dot{\gamma}(t) = [A(t,y,u) + lc]\gamma(t)$ is exponentially stable, then the following adaptive observer is possible:

$$\dot{\hat{x}} = [A(t,y,u)+lc]\hat{x}+lce+bu+f(y,u,t)+p(t,y,u)\phi+\gamma(t)\dot{\phi}, \; e := \hat{x}-x,$$

$$\dot{\phi} = -\gamma^\top(t)c^\top ce$$

$$\dot{\gamma}(t) = [A(t,y,u) + lc]\gamma(t) + p(t,y,u),$$

provided $\gamma(t)$ is PE.

No realistic test for PE exists.

G. Besanon, J. D. Leon-Morales, and O. Huerta-Guevara, "On adaptive observers for state affine systems", Int. J. Control, vol. 79, no. 6, pp. 581-591, 2006.

X. Liang, J. Zhang, and X. Xia, "Adaptive Synchronization for Generalized Lorenz Systems", IEEE Trans. Automat. Contr., 53 (2008), No. 7, pp. 1740–1746.

Adaptive observers
Nonmeasurable nonlinearities

$$\dot{x} = Ax + f(x,u,t) + p\,\beta(x,u,t)\delta, \quad y = cx, \quad c = [1,0,\dots,0],$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \delta = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_q \end{bmatrix} \in R^q, p = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \in R^n.$$

Static adaptive observers using LMI condition.

Static adaptive high-gain observers are not possible even for strictly triangular systems.

High-gain local dynamic adaptive observers possible for strictly triangular nonlinearities (Celikovsky, Torres, Rodriguez, Dominguez-Bocanegra, 2015).

Dynamic adaptive observers
Nonmeasurable nonlinearities

$$A(r) = \begin{bmatrix} rl_1 & 1 & 0 & \cdots & 0 \\ r^2 l_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ r^{n-1} l_{n-1} & 0 & \cdots & 0 & 1 \\ r^n l_n & 0 & \cdots & 0 & 0 \end{bmatrix},$$

Dynamic adaptive observers

Nonmeasurable nonlinearities

Theorem

Let $F(z), G(z), P(z)$ be Jacobians of $f(z), g(z), p(z)$ at $z \in R$. Then, $\forall l_1, \ldots, l_n$ such that the matrix A_l is Hurwitz and $\forall k > 0$ there exists $r > 1$ and $\delta_0 > 0$ s.t. $\forall \phi_0 \in (\delta - \delta_0, \delta + \delta_0)$

$$\begin{aligned} \dot{z} &= Az + f(z) + g(z)u + [l_1 r, \ldots, l_n r^n]^\top z_1 + p(z)\phi + \gamma(t)\dot{\phi} \\ \dot{\phi} &= -k\gamma^\top(t)c^\top ce, \quad \phi(0) = \phi_0, \\ \dot{\gamma} &= [A(r) + F(z) + G(z)u + P(z)\phi]\gamma(t) + p(z) \end{aligned}$$

is the adaptive exponential observer provided that the signal $\gamma^\top(t)c^\top$ is the persistently exciting one.

Wastewater monitoring
Wastewater plant model

Model of bioreactor containing a continuous culture of *Spirulina maxima*. Denote the biomass x and the substrate s::

$$\left.\begin{array}{r} \dot{x} = x\mu(s) - x(u+\delta) \\ \dot{s} = -a_3^{-1}x\mu(s) + (a_4 - s)(u+\delta) \end{array}\right\}$$

Monod's rate of growth $\mu(s) = \frac{a_1 s}{a_2 + s}$.
a_1 is the maximal rate of growth, a_2 represents the Monod's saturation constant, a_3 is the yield coefficient, a_4 is the input substrate concentration.
The control input: the rate of dilution $u(t)$ feeding the bioreactor. It is biased by the constant unknown dilution supplement $\delta > 0$.

Note, that model structure does not fit even the most complex case of previous overview!!!

Wastewater monitoring

Transformation to new coordinates

$$x_1 := \ln(x),\ x_2 := \mu(s), \quad x = \exp(x_1),\ s = \frac{a_2 x_2}{a_1 - x_2},$$

$x_1 \in R$, $\exists\forall x > 0$, $x_2 \in [0, a_1]$, $\exists\forall s > 0$. Transformed equations:

$$\begin{aligned} \dot{x}_1 &= x_2 - u - \delta \\ \dot{x}_2 &= \frac{x_2 \exp(x_1) a_1 a_2}{-a_3(a_2 + \frac{a_2 x_2}{a_1 - x_2})^2} + (a_4 - \frac{a_2 x_2}{a_1 - x_2}) \frac{a_1 a_2}{(a_2 + \frac{a_2 x_2}{a_1 - x_2})^2}(u + \delta). \end{aligned}$$

$$\begin{aligned} \dot{x}_1 &= x_2 - u - \delta \\ \dot{x}_2 &= -a_1^{-1} a_2^{-1} a_3^{-1} \exp(x_1) x_2 (a_1 - x_2)^2 + \\ &\quad a_1^{-1} a_2^{-1} (a_4 a_1 - (a_2 + a_4) x_2)(a_1 - x_2)(u + \delta). \end{aligned}$$

Avoids singularities. Right hand side is not globally Lipschitz on the whole, but it is globally Lipschitz on bounded forward invariant subset of R^2.

Wastewater monitoring

Biological consistency of the model

Proposition

Consider the system

$$\dot{x} = x\mu(s) - x(u+\delta), \; \dot{s} = -a_3^{-1} x\mu(s) + (a_4 - s)(u+\delta),$$

where $\mu(s) = \frac{a_1 s}{a_2 + a_1 s}$. *The following properties hold:*
(i) The set given by $\mathcal{O}^+ := \{x > 0, \; a_4 > s > 0\}$ *is forward invariant for every integrable input signal* $u(t)$ *and constant perturbation* δ.
(ii) Every solution starting in $\mathcal{O}^+ := \{x > 0, \; a_4 > s > 0\}$ *is bounded for every integrable input signal* $u(t)$ *and perturbation* δ.

For practical reasons, after some time $s(t) < a_4$, since a_4 is the input substrate concentration.

Wastewater monitoring

Biological consistency of the model

Proof.

To prove (i), check the right hand side on the boundary of $\mathcal{O}^+$. As $x = 0$ is invariant, $x(0) > 0 \Rightarrow x(t) > 0 \; \forall t \geq 0$. Next, on the set $s = 0$ one has $\dot{s} \geq 0$ while on $s = a_4$ it holds $\dot{s} \leq 0$.

To prove (ii), introduce $\tilde{x} = x + a_3 s - a_3 a_4$, $\tilde{s} = s$, giving

$$\dot{\tilde{x}} = -(u + \delta)\tilde{x}, \dot{\tilde{s}} = -a_3^{-1}(\tilde{x} - a_3\tilde{s} + a_3 a_4)\mu(\tilde{s}) + (a_4 - \tilde{s})(u + \delta).$$

Consider $V = \frac{1}{2}\tilde{x}^2$, then $\dot{V} = -(u + \delta)\tilde{x}^2$. Therefore, trajectory starting at the point $\tilde{x}_0, \tilde{s}_0$ for $t = 0$ stays for $t \geq 0$ inside the set where $|\tilde{x}| \leq \tilde{x}_0$ for every integrable input and perturbation signals. In the original coordinates: the strip formed by two lines parallel with the line $x + sa_3 = 0$, its intersection with positive orthant is bounded for any $a_3 > 0$ by (i). □

Wastewater monitoring

Adaptive observer with dynamical gains

We have put wastewater model into the form

$$\begin{aligned}\dot{x}_1 &= x_2 + g_1(x)(u+\delta)\\ \dot{x}_2 &= f_2(x) + g_2(x)(u+\delta)\end{aligned}$$

$$\begin{aligned}f_1(x) &= 0; \quad f_2(x) = -a_1^{-1}a_2^{-1}a_3^{-1}\exp(x_1)x_2(a_1-x_2)^2;\\ g_1(x) &= -1; \quad g_2(x) = a_1^{-1}a_2^{-1}(a_4a_1-(a_2+a_4)x_2)(a_1-x_2).\end{aligned}$$

By (Celikovsky, Torres, Rodriguez, Dominguez-Bocanegra, 2015)

$$\begin{aligned}\dot{z}_1 &= z_2 + g_1(z)(u+\phi) + rl_1(z_1-x_1) + \gamma_1(t)\dot{\phi}\\ \dot{z}_2 &= f_2(z) + g_2(z)(u+\phi) + r^2l_2(z_1-x_1) + \gamma_2(t)\dot{\phi},\end{aligned}$$

$$\begin{aligned}\dot{\phi} &= -\kappa\gamma_1(t)(z-x), \quad \kappa>0,\\ \dot{\gamma}_1 &= rl_1\gamma_1 + \gamma_2 + p_1(z),\\ \dot{\gamma}_2 &= r^2l_2\gamma_1 + F_{21}(z)\gamma_1 + F_{22}(z)\gamma_2 + G_{22}(z)(u+\phi)\gamma_2 + p_2(z),\end{aligned}$$

is the dynamic adaptive observer. Here

Wastewater monitoring

Adaptive observer with dynamical gains

$$
\begin{aligned}
p_1(z) &= g_1(z) = -1 \\
p_2(z) &= g_2(z) = a_1^{-1} a_2^{-1} (a_4 a_1 - (a_2 + a_4) z_2)(a_1 - z_2) \\
F_{21}(z) &= \frac{\partial}{\partial z_1} f_2(z) = -a_1^{-1} a_2^{-1} a_3^{-1} \exp(z_1) z_2 (a_1 - z_2)^2 \\
F_{22}(z) &= \frac{\partial}{\partial z_2} f_2(z) = -a_1^{-1} a_2^{-1} a_3^{-1} \exp(z_1)(a_1 - z_2)(a_1 - 3z_2) \\
G_{22}(z) &= \frac{\partial}{\partial z_2} g_2(z) = -2 a_1^{-1} a_2^{-1} \left[a_4 a_1 - (a_2 + a_4) z_2 \right] - 1.
\end{aligned}
$$

Wastewater monitoring

LMI based static adaptive observer

Result has been just submitted to ACC 2016 by J. Torres, A. Sonck, S. Celikovsky and A.R. Dominguez. Consider again

$$\dot{x} = x\mu(s) - x(u+\delta), \;\; \dot{s} = -a_3^{-1} x\mu(s) + (a_4 - s)(u+\delta),$$

$$\mu(s) = \frac{a_1 s}{a_2 + a_1 s}, \;\; \frac{\mathrm{d}\mu(s)}{\mathrm{d}s} = \frac{a_1 a_2}{(a_2+s)^2}$$

Yet another transformation!

$$x_1 = \log(x), \;\; x_2 = \log(a_4 - s), \;\; x = \exp(x_1), \;\; s = a_4 - \exp(x_2).$$

$$\dot{x}_1 = \frac{a_1 a_4 - a_1 \exp(x_2)}{a_2 + a_1 a_4 - a_1 \exp(x_2)} - (u+\delta),$$

$$\dot{x}_2 = \frac{-\exp(x_1)}{a_3} \frac{a_1 a_4 - a_1 \exp(x_2)}{a_2 + a_1 a_4 - a_1 \exp(x_2)} + (u+\delta)$$

Wastewater monitoring

LMI based static adaptive observer

Previous equations can be rewritten as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_2) \\ f_1(x_2)f_2(x_1) \end{bmatrix} + (u+\delta) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$f_1(x_2) = \frac{a_1 a_4 - a_1 \exp(x_2)}{a_2 + a_1 a_4 - a_1 \exp(x_2)}, \; f_2(x_1) = \frac{-\exp(x_1)}{a_3}.$$

Assume now, moreover, that $s \leq a_4 - \alpha$ for some $\alpha > 0$, then

$$f_1'(x_2) = \frac{-a_1 a_2 \exp(x_2)}{[a_2 + a_1 a_4 - a_1 \exp(x_2)]^2} \;\Rightarrow\; -\frac{a_1 a_4}{a_2} \leq f_1'(x_2) \leq -\alpha$$

Static observer:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} f_1(z_2) \\ f_1(z_2)f_2(x_1) \end{bmatrix} + (u+\phi) \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} (z_1 - x_1)$$

$$\dot{\phi} = -\beta(z_1 - x_1), \beta > 0.$$

Wastewater monitoring
LMI based static adaptive observer

Error dynamics

$$\dot{e} = \begin{bmatrix} l_1 & f(t) \\ l_2 & f(t)f_2(x_1) \end{bmatrix} e + \epsilon \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\dot{\epsilon} = \beta c e, \quad c = [1, 0], \quad f(t) = f_1'(x_2(t) + \theta(z(t) - x(t))), \ \theta \in [0, 1].$$

$$-\frac{a_1 a_4}{a_2} \leq f(t) \leq -\alpha,$$

$$0 > f_2(x_1) \geq -M_x / a_3 := B.$$

Here, M_x is some a priori bound of values of variable x.
Next, find $P, l = [l_1, l_2]^\top$ such that:

$$\begin{bmatrix} l_1 & f_{12} \\ l_2 & f_{22} \end{bmatrix}^\top P + P \begin{bmatrix} l_1 & f_{12} \\ l_2 & f_{22} \end{bmatrix} < 0, f_{12} \in \{\frac{a_1 a_4}{-a_2}, -\alpha\}, f_{22} \in \{0, \frac{a_1 a_4}{a_2} B\},$$

$$P \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad P = P^\top, \qquad P > 0.$$

Wastewater monitoring

LMI based static adaptive observer

In total, there are 2 matrix equalities and 5 matrix inequalities to be solved by variables P, l, so using 6 real scalar variables. Tight, but feasible solutions were P, l successfully obtained!

Remark: The problem becomes LMI only after coordinate change $X = P, Y = Pl$, which is the well-known procedure.

Having P, l solving the above equalities and inequalities, Lyapunov-like function $V = (\beta/2)e^{\top}Pe + (1/2)\epsilon^2$ shows that $e \to 0$ as $t \to \infty$. LaSalle principle can be used to show the identification as well and β is an extra design parameter to adjust speed of convergence of unknown parameter estimate.

Finally, numerical experiments show that using high $\beta > 0$ allows to handle even time varying or state dependent uncertainties as well, but giving "practical" adaptive observers only.