EVROPSKÁ UNIE

MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

OP Vzdělávání pro konkurenceschopnost

VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ

INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Vybrané partie z teorie automatického řízení lineárních systémů

Vladimír Kučera **Jiří Cigler**

14. ledna 2010

Tato prezentace je spolufinancována Evropským sociálním fondem a státním rozpočtem České republiky.

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Feedback Control: the Origins, the Milestones, and the Trends

Vladimír Kučera

14. ledna 2010

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Feedback in the Nature

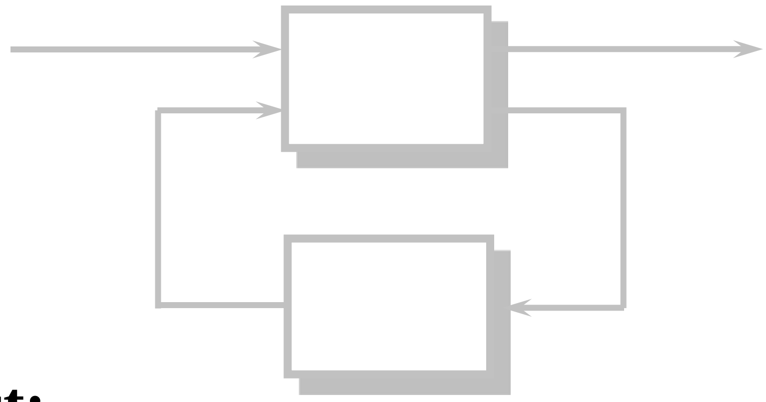
Feedback control is the basic mechanism by which systems, whether mechanical, electrical, biological, economic or social, maintain their equilibrium.

The conditions under which life can continue are narrow.
A change in body temperature of half a degree
is generally a sign of ilness.
The homeostasis of the body is maintained
through the use of feedback control.

Feedback over long time periods is responsible
for the evolution of species.

Definition of Feedback Control

Feedback control may be defined
as the use of difference signals,
determined by comparing the actual values
of system variables to their desired values,
as a means of controlling a system.



The concept of feedback is abstract;
it is not tied to any particular signal or system.

Man-Made Control Systems

Examples of man-made systems abound.

Feedback control is used by engineers so that
the temperature in our homes stays within acceptable levels;
airplanes maintain desired heading, speed and altitude;
automobile emissions meet specifications.

In biomedical applications,
electrical nerve signals are used to control prosthetics;
robots cut holes in bone for implanting artificial joints.

Economic indicators, such as unemployment and inflation,
are controlled by government fiscal decisions.

Feedback Control
is an Engineering Discipline

As such, its progress is closely tied to the practical problems that needed to be solved during any phase of human history.

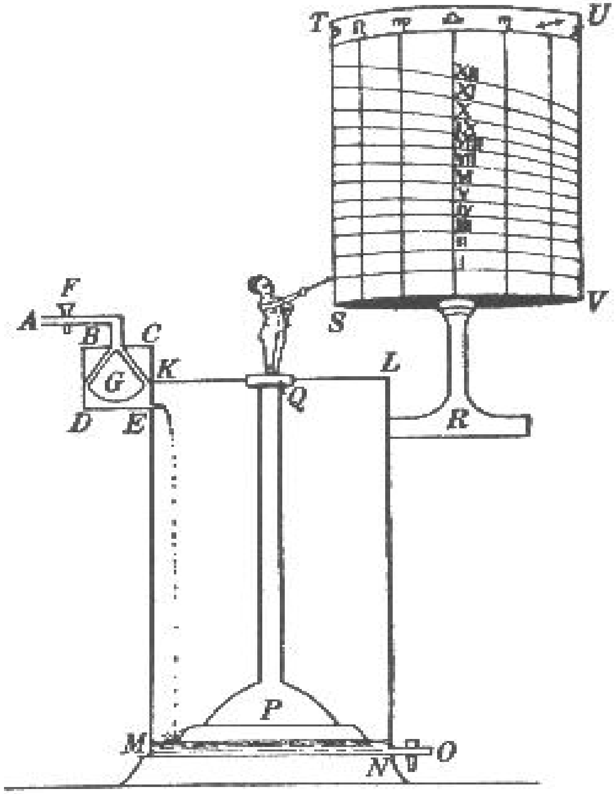
The key developments in the history of mankind that affected the progress of feedback control were

- **the preoccupation of the Greeks and Arabs with keeping accurate track of time;**
- **the Industrial Revolution in Europe;**
- **the beginning of mass communication;**
- **the beginning of space/computer age.**

Water Clock of Ktesibios

The first control device on record;

Alexandria around 270 B.C.



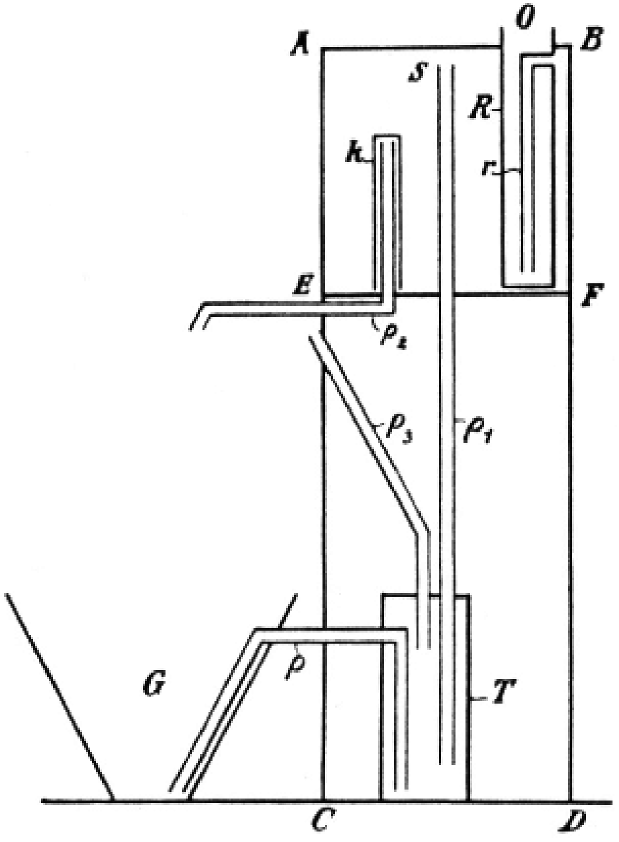
Gave rise to

float valve regulators

Water Supply Device

The three brothers Banū Mūsā,

Baghdad around 850 A.D.



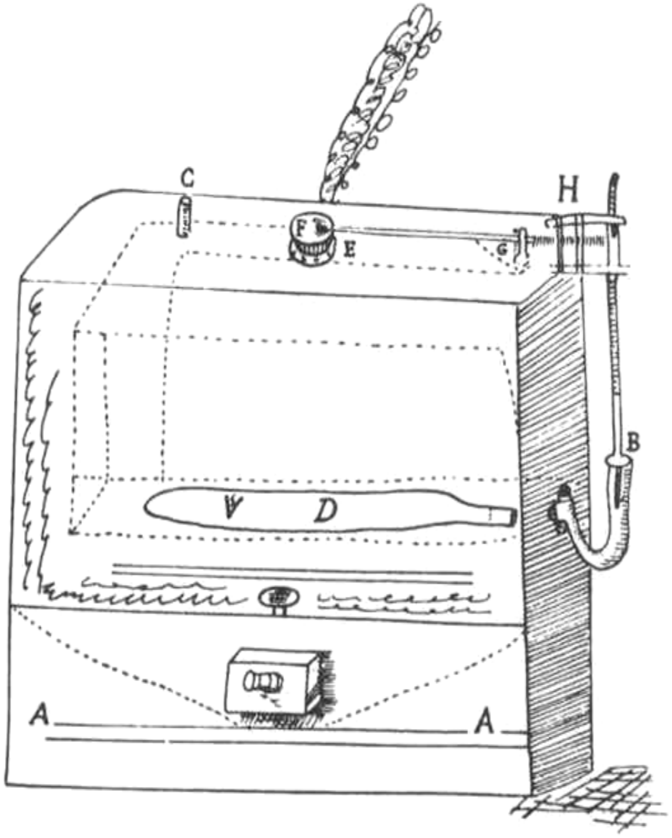
Gave rise to

on-off control systems

Incubator of Drebbel

Holland

around 1620



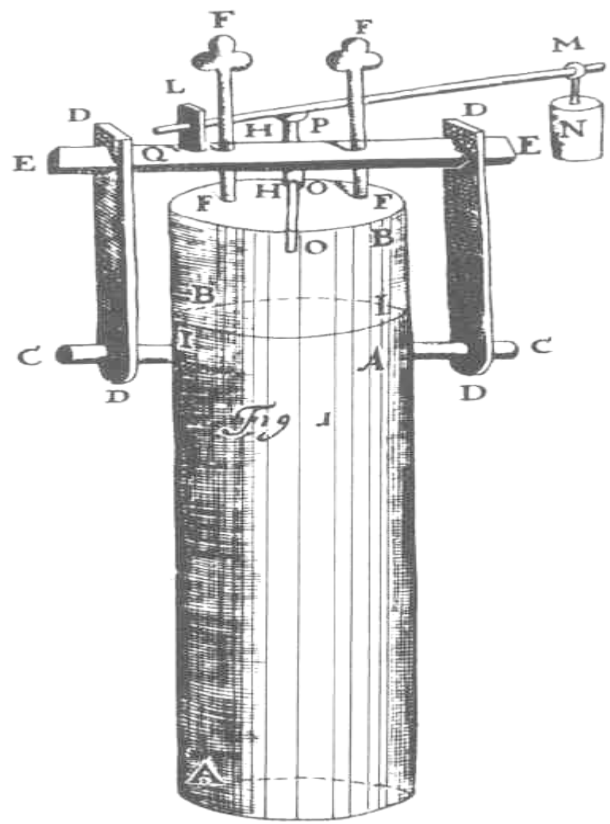
Gave rise to

temperature regulators

Pressure Cooker of Papin

France

in 1681



Gave rise to

safety valves

Centrifugal Governor of Watt

By far the most famous control device,

England in 1788

Impressive demonstration

of the action of feedback

From Intuition to Science

- **Stability analysis**

  Maxwell 1868, Lyapunov 1892

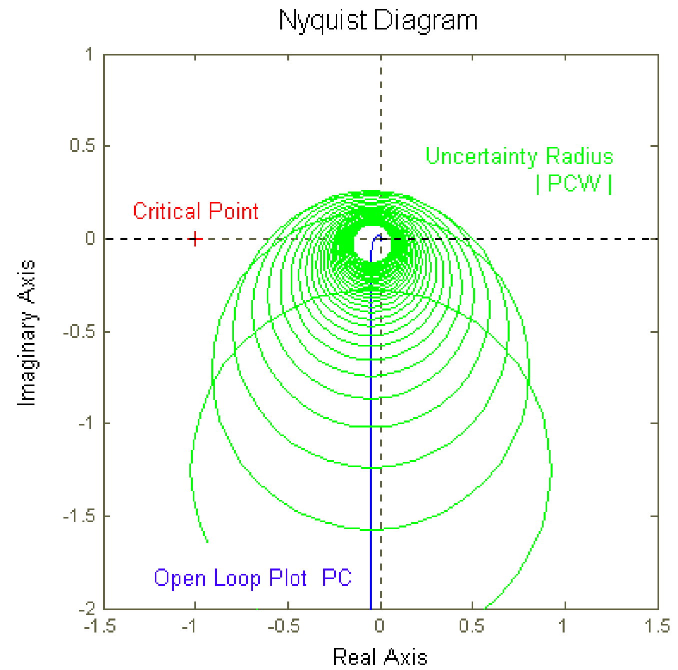
- **Feedback amplifiers**

  Black 1927, Nyquist 1932

- **PID Controllers**

  Minorsky 1922,

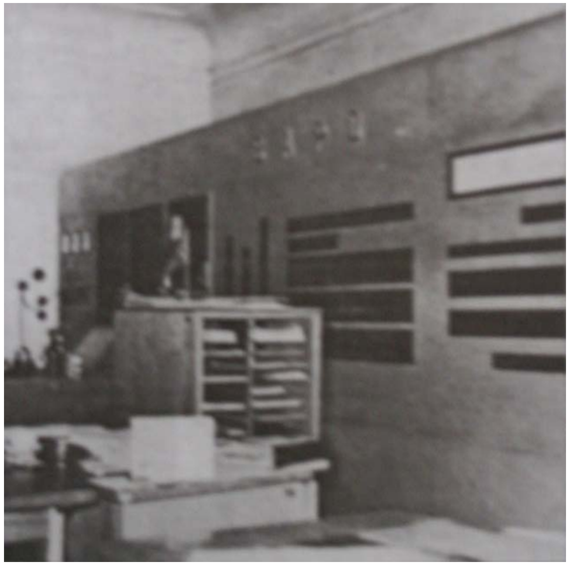
  Ziegler & Nichols 1942



Control theory based on mathematical models; control devices replaced by universal controllers, making no longer an integral part of the inventions.

The Development and Impact of Computers

From a stored-program machine to the microcomputer

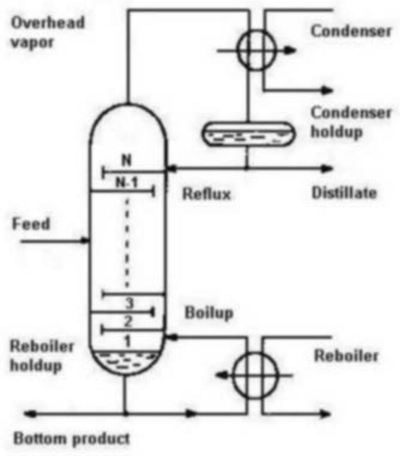


- **Methods**
  - system theory, information science
  - optimal and robust control
  - numerical techniques
- **Implementation**
  - digital controllers
  - smart sensors and actuators

Computer technology represents the most significant revolution of the 20th century.

The Importance of Applications

The truly exciting developments in any field will occur where there is a confluence of application drivers and disciplinary development of the subject.



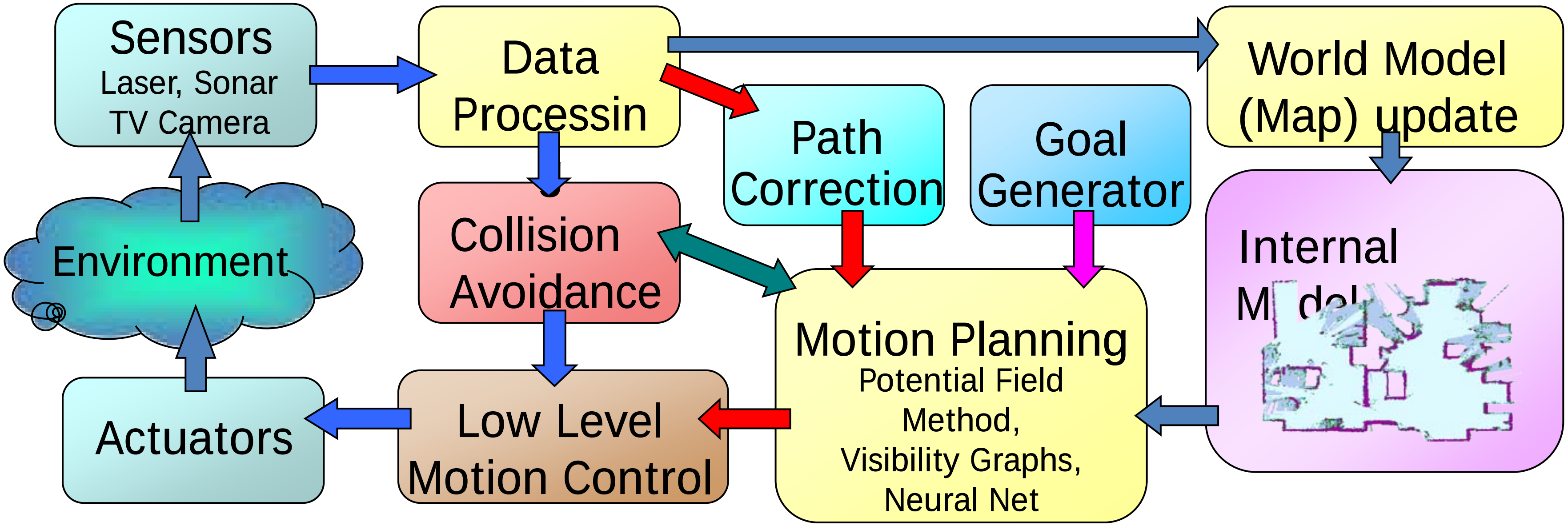
Automatic control is no exception.

Nowadays, technology seems to advance faster than theory.

Example: Mobile Robot

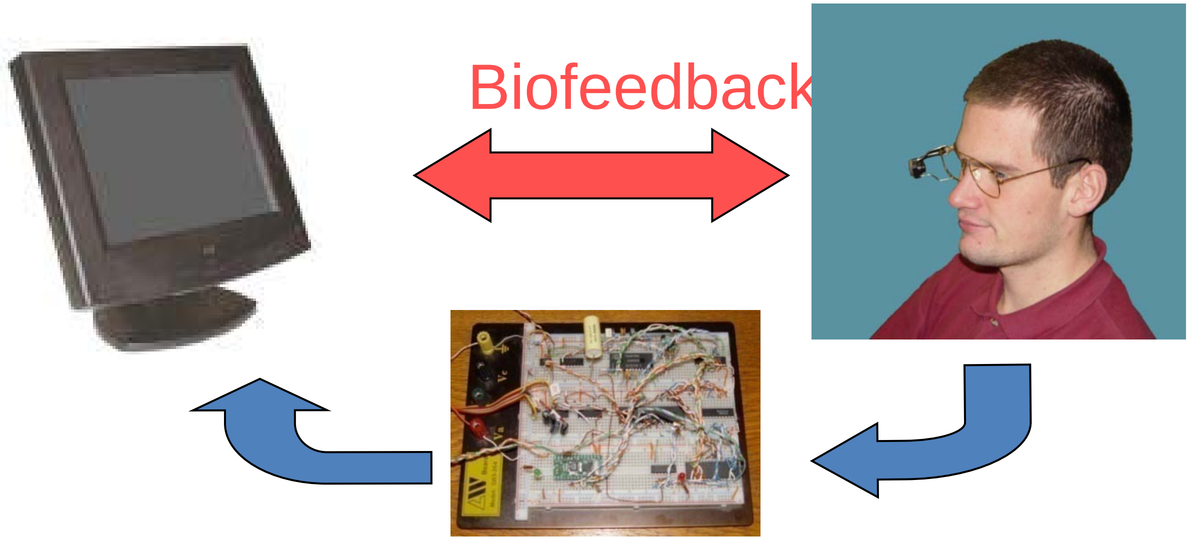
Autonomous vehicle, multilevel feedback

- Interaction with scene obstacles
- Trajectory execution and control
- World-model update



Example:
Mobile Eye Movements Recorder

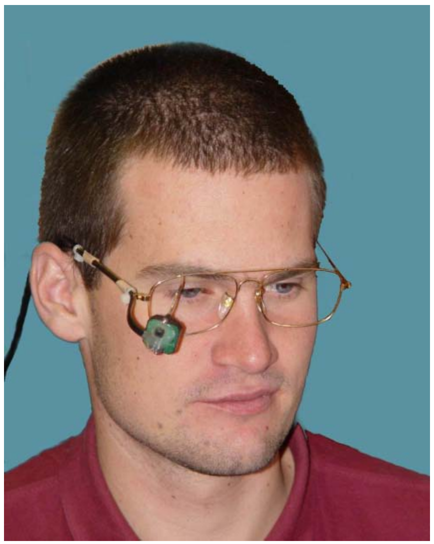
A small camera and a communication module simulate a mouse, thus enabling a handicapped person to control a PC by eye movements.

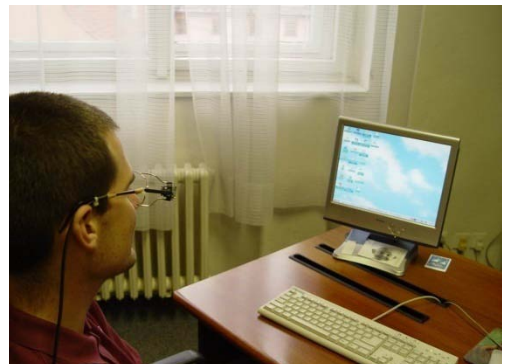


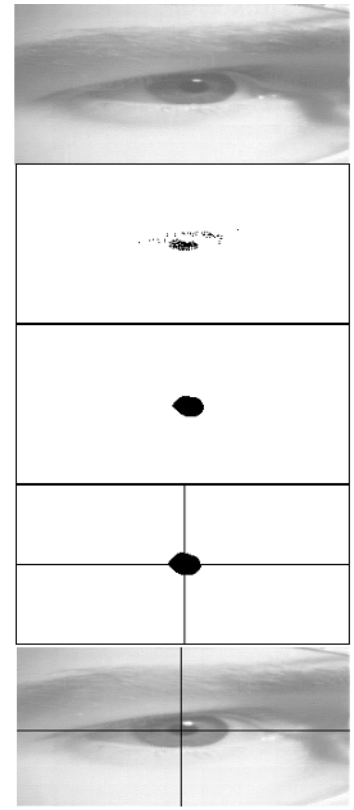
Feedback control using eye movements.

Rehabilitation of Paralytic Strabismus

Abnormal alignment of eyes, either inwards or outwards, which prevents parallel vision.







laboratory Gerstner

Example: Paraplegics

Restore lost function to paralyzed individuals
by using feedback control of electrical stimulation systems.

paraplegic standing

paraplegic cycling

Polynomial methods
The University of Glasgow

Example: Rescue Robot

VUT Brno

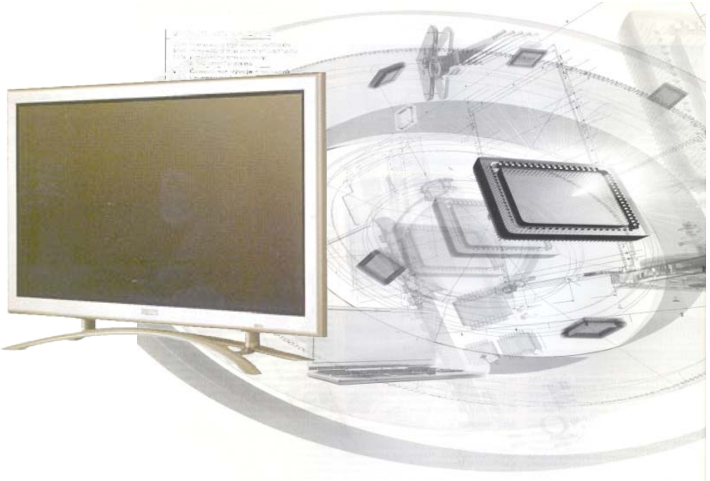
World champions

RoboCup Padova 2003

Significant Trends …

- Performance improves
- Costs decline
- Reliability and safety increase
- Energy consumption decreases
- Impossible becomes do-able

as demonstrated by
automotive applications,
process control and robotics,
semiconductor processing,
hard disc drives,
and numerous consumer products.



… and Their Facilitators

- advanced control methods

  optimal, robust, and adaptive control

- smart sensors

  communicate, self diagnose, or make decisions

- analog controllers replaced by digital ones

  more efficient, less sensitive, immune to aging,

  more precise, programmable

- smart actuators

  monitor, diagnose, and optimize

  to more faithfully produce desired outputs

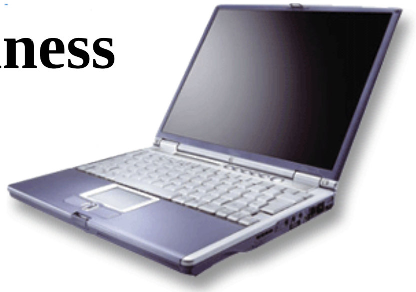
Emerging Methods and Technologies

- Computers and information technology
  embedded computer functions
- Networking and communication
  distributed control over networks
- Nanoscale science
  compact intelligent components
- Transportation and vehicles
  performance and safety
- Manufacturing and processes
  plant wide automation
- Medicine and biology
  biosensors and biocomputers

Future Developments

Computers and information technology

- **ubiquitous availability of computers will facilitate continuous monitoring and automatic control**
- **computer functions embedded in everyday functions, new products must be conceived holistically**
- **computer technology will enable many new consumer products and foster new perspectives for work and business**



Subsystems will be integrated and embedded to yield improved overall systems.

Future Developments

Networking and communication

- computers and communications
  will become indistinguishable
  and their integration will become a core utility
- embedded functions and networking
  will result in systems of bewildering complexity
- increased development of distributed control techniques,
  applications using wireless communication technology

The worldwide broadband network will prove to be the most significant revolution of the 21st century.

Future Developments

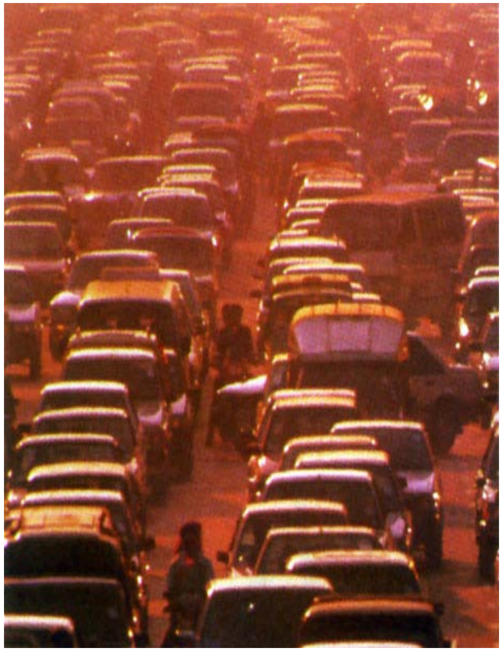
Nanoscale science

- **nanotechnology, the coming wave of miniaturization, is taking shape at the intersection of chemistry, physics, biology, and electrical engineering**
- **nanotechnology will create and utilize materials, devices and systems with novel properties achieved through the control of matter atom by atom**
- **nanotechnology will leverage development of innovative, ubiquitous sensors and actuators**

Nanotechnology underpins innovation in information technologies and manufacturing.

Future Developments

Transportation and vehicles



- **high traffic density,**
  safety situation still unsatisfactory
- **advanced driver assistance systems**
  will help to focus on critical situations
- **when combined**
  with infrastructure improvements
  will yield intelligent traffic control

These developments will also leverage developments
in autonomous unmanned vehicles
for situations such as operation in hostile environments.

Future Developments

Manufacturing and processes

- **future manufacturing will be highly automated, unmanned factories and processing plants**
- **plant wide automation will take engineering as well as economic and environmental criteria into account**
- **hybrid control systems will be studied to meet these requirements**

Feedback will be used mostly
to stabilize the process and to counteract uncertainties,
other functions achieved by a feedforward.

Trends in Control Theory

Analytic methods give way to computer-based solutions

Problems considered solved when …

$$P=\frac{B}{A}$$

- **closed-form solutions found (1940s)**
  then used in both analysis and synthesis
  (Wiener formulae and optimal filtering)

$$C=\frac{Y-AW}{X+BW}$$

- **governing equations derived (1960s)**
  and an algorithmic solution left to a computer
  (Riccati equations and optimal control)

$$AX+BY=1$$

- **mathematical programs set up (1990s)**
  with much of the analysis replaced by computer-aided design
  (linear and semidefinite programs and robust control)

Future Challenges and Opportunities

Control theory is looking for new solutions
to address the control of complex systems, or systems of systems

Focused interdisciplinary research

- **distributed control over communication networks**
- **real-time control and re-configurable systems**
- **hybrid control systems**
- **collaborative control/ agent based models**
- **links with life sciences**
- **managing complexity**

Strong influence of computer science,
model-based as well as knowledge-based methods.

Control Under Communication Constraints

The goal is to design sensors, encoders, communication channels and controllers so as to achieve prescribed performances despite of all the constraints and obstacles imposed by the communication channels and in the presence of possible uncertainties and disturbances

Constraints imposed by the communication channels include

- **bandwidth**
- **delays of variable amount**
- **quantization errors**
- **transmission noise**
- **random loss of information**

Collaborative Control

Collaborative control trends are apparent in all types of distributed/ hierarchical systems

Machine - Machine:

- **robotic teams will be able to interact better than human teams**
- **control methods to manage faults, conflicts, and interactions**

Human - Machine:

- **better understanding of how to share tasks**
- **new sensors and actuators tailored for human use**

Human - Human:

- **enterprise software integrates and aids team decisions**
- **internet conferencing improves team coordination**

Links with Life Sciences

The government problem: increased life expectancy and payment for health care.

The biotechnology revolution, which is forcing many biologists to understand issues of feedback modeling and system theory.

Control, communications and computer devices in medicine.

Biological structures and capabilities may inspire mimicry.

Medical interventions may first require identification of very complex systems.

Nature has not done analytic design; it has produced brilliant iterative design. How do we shift our thinking to do the same?

Managing Complexity

The complexity is overwhelming the designers and managers.

- **Communication system complexity**
  scalability, adaptivity, decentralized architecture
- **Power system complexity**
  interconnections, large scale, stability, big money
- **The national economy**
  part of the world economy, many agents
  humans in the economic loop, complicating control
  billions euros/crowns at stake
  livelihoods of families at stake

What is the right architecture
for sensing, communication and control?

Educated Workforce

- Need for individuals
  who are educated to unprecedented levels
  of scientific and technological expertise

- Multidisciplinary education with emphasis
  on generic methods rather than vocational skills

- Higher education more accessible
  due to the development of distance learning opportunities

- Life long learning will be the norm

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Polynomial Equation Approach to Control System Design

Vladimír Kučera

14. ledna 2010

Tato prezentace je spolufinancována Evropským sociálním fondem a státním rozpočtem České republiky.

Outline of the Presentation

Basic tools:
parameterization of all stabilizing controllers
linear equations for polynomials (Diophantine equations)

Motivation, historical notes

Standard applications:
asymptotic properties, pole placement, deadbeat control,
H_2 optimal control, l_1 optimal control, robust control

Advanced applications:
stabilization subject to input constraints,
input and output shaping, fixed-order controller design

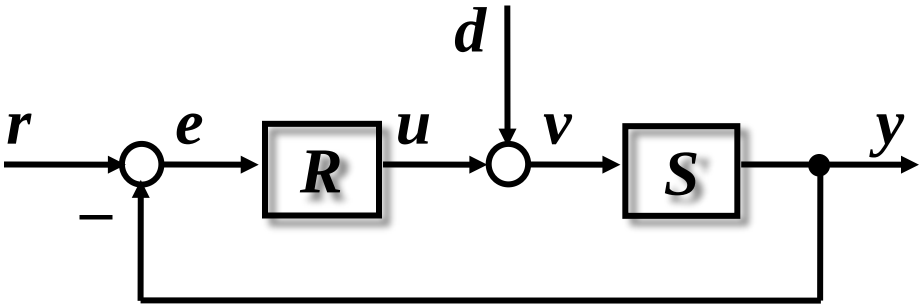
A Typical Control Problem

Given a plant *S*,

determine a controller *R* so that

(1) the control system is stable, either Re*s* < 0 or |*z*| < 1

(2) additional specifications are met.



It is logical to stabilize first,

then meet the additional specifications.

Then one needs to determine *all* stabilizing controllers.

Polynomial Description

Let $S = b/a$ and $R = q/p$, coprime polynomial fractions.

Closed loop sensitivity

$$H_S = \frac{1}{1+SR} = a\frac{p}{ap+bq} := aX$$

and complementary sensitivity

$$H_C = \frac{SR}{1+SR} = b\frac{q}{ap+bq} := bY$$

In a stable system, X and Y are stable.

However, X and Y cannot be arbitrary since $H_S + H_C = 1$.

Hence

$$aX + bY = 1$$

Parameterization of Stabilizing Controllers

All controllers that stabilize the plant $S = b/a$ are given by $R = Y/X$, where X, Y is a *stable rational* solution pair of

$$aX + bY = 1$$

All solution pairs can be expressed in parametric form as

$$X = x + bW, \quad Y = y - aW$$

where x, y are polynomials such that $ax + by = 1$ and W is a free *stable rational* parameter.

This is a fundamental result, called the *Youla-Kučera parameterization*.

Example 1

Plant

$$S(s)=\frac{1}{s}$$

Equation

$$sx+y=1$$

A solution $x = 0$, $y = 1$ yields the stabilizing controllers

$$R(s)=\frac{1-sW}{W},\quad W\neq 0 \text{ stable rational}$$

For example, $W = 1/(s+1)$ yields a proportional controller $R = 1$.

Taking $W = 1$ results in $R(s) = 1 - s$;
this controller is stabilizing but it is not proper
and the feedback system has a pole at $s = \infty$.

Dicsrete-Time Systems

The parameterization applies to discrete-time systems as well.

Continuous-time systems can give rise to transfer functions that are not proper.

In the case of discrete-time systems, however,
additional constraints have to be imposed:
the transfer functions *S* and *R* are to be *proper*
(so that the plant and the controller are causal systems)
and one of them is to be *strictly proper*
(so that the closed loop system is causal).
The chronology of samples in the control system
is usually taken in such a way that *S* is strictly proper.

Example 2

Plant

$$S(z) = \frac{1}{z-1}$$

Write

$$S(z) = \frac{z^{-1}}{1-z^{-1}}$$

Equation

$$(1-z^{-1})x + z^{-1}y = 1$$

A solution $x = 1$, $y = 1$ yields the stabilizing controllers

$$R(z) = \frac{1-(1-z^{-1})W}{1+z^{-1}W}$$

for any proper stable rational W.

Historical Notes

Jury 1959	**deadbeat, SISO plant**
Volgin 1962	**pole placement**
Åström 1970	**minimum variance, minimum phase plant**
Peterka 1972	**minimum variance**
Kučera 1973	**stabilization, parameterization, SISO plant**
Kučera 1975	**stabilization, parameterization**
Youla et al 1976	H_2 **control, stabilization, parameterization**
Kučera 1979	**polynomial equation approach**
Desoer et al 1980	**proper stable fractions**
Nett et al 1984	**state space formulas**

It took decades to appreciate the importance of the result and come up with applications.

Additional Performance Specifications

- There are as many stabilizing controllers for a given plant as stable rational free parameters *W*.
- The set of stabilizing controllers for a given plant contains controllers of arbitrarily high order.
- The parameter *W* in turn parameterizes all resulting stable closed-loop transfer functions and the parameterization is *linear* in *W*,

$$\begin{bmatrix} v \\ y \end{bmatrix} = \frac{1}{1+SR}\begin{bmatrix} 1 & R \\ S & SR \end{bmatrix}\begin{bmatrix} d \\ r \end{bmatrix} = \begin{bmatrix} a(x+bW) & a(y-aW) \\ b(x+bW) & b(y-aW) \end{bmatrix}\begin{bmatrix} d \\ r \end{bmatrix}$$

while it is *nonlinear* in *R*.

Asymptotic Properties

Reference tracking

output y follows reference r (error e goes to zero) asymptotically

In terms of Laplace transforms,

$e(s) = H_S(s)\, r(s)$ is to be a stable rational function.

Disturbance attenuation

effect of disturbance d on output y decreases asymptotically

In terms of Laplace transforms,

$y(s) = SH_S(s)\, d(s)$ is to be a stable rational function.

This is to be achieved by a selection of the parameter W.

Example 3

Plant

$$S(s) = \frac{1}{s+1}$$

Stabilizing controllers

$$R(s) = \frac{1-(s+1)W}{W}$$

for any stable rational $W \neq 0$.

Achievable sensitivity transfer functions are $H_S = (s + 1)W$.

To track a step reference, $r(s) = 1/s$, we must take $W = sW_1$ for any stable rational $W_1 \neq 0$.

To attenuate a sinusoidal disturbance, $d(s) = s/(s^2 + \omega^2)$, we constrain the parameter as $W = (s^2 + \omega^2)W_2$ for any stable rational $W_2 \neq 0$.

Demonstrates the *internal model principle*.

Pole Placement

Plant $S = b/a$

Stabilizing controller $R = Y/X$,

where $X = x + bW$, $Y = y - aW$ **and** $ax + by = 1$.

Let $W = w/d$, **where** d **is a Hurwitz polynomial.**

Then

$$R = \frac{dy - aw}{dx + bw} := \frac{q}{p}$$

Pole placement equation

$$ap + bq = d(ax + by) = d$$

The polynomial d **specifies the closed-loop poles while** w **represents the remaining degrees of freedom.**

Example 4

Plant

$$S(s) = \frac{1}{s-1}$$

Stabilizing controllers

$$R(s) = \frac{1-(s-1)W}{W} \quad , W \neq 0 \text{ stable rational}$$

Let the desired pole locations be given by $d(s) = s^2 + 2s + 1$.

Put $W = w/d$.

Then

$$R(s) = \frac{(s^2 + 2s + 1) - (s-1)w}{w}$$

and for R to have order 1, take $w(s) = s + \omega$ for any real ω.

Otherwise poles at $s = \infty$ as well.

Deadbeat Control

A *discrete-time* control problem

Plant $S = b/a$

Find a stabilizing controller $R = Y/X$

such that all four closed- loop transfer functions

$$H_S = a\,(x + bW),\quad SH_S = b(x + bW),$$

$$H_C = b(y - aW),\ S^{-1}H_C = a(y - aW)$$

are FIR (vanish in a finite/ shortest time).

This occurs iff *W* is a *polynomial* in z^{-1}**.**

Special case of pole placement: all poles at $z = 0$**.**

Shortest transient time iff

x, y **is the least-degree solution pair of** $ax + by = 1$**.**

Example 5

Plant

$$S(z) = \frac{z^{-1}}{1 - z^{-1}}$$

Stabilizing controllers

$$R(z) = \frac{1 - (1 - z^{-1})W}{1 + z^{-1}W}$$

Then

$$H_S = 1 - z^{-1} + z^{-1}(1 - z^{-1})W, \quad SH_S = z^{-1} + z^{-2}W,$$

$$H_C = z^{-1} - z^{-1}(1 - z^{-1})W, \quad S^{-1}H_C = 1 - z^{-1} - (1 - z^{-1})^2 W$$

are all polynomials in z^{-1} iff W is a polynomial in z^{-1}.

The shortest impulse responses are achieved for $W = 0$.

The transients will vanish in one step.

H_2 Optimal Control

Plant $S = b/a$

Find a stabilizing controller $R = Y/X$

such that, say, $H_C = b(y - aW)$ has a least H_2 norm.

Let $\alpha\beta$ be a polynomial

defined by keeping the stable (in Res < 0) zeros of ab

while replacing the unstable (in Res ≥ 0) ones

with their negative values.

In fact, α is the spectral factor of $a(s)a(-s)$, β is that of $b(s)b(-s)$.

Then $ab/\alpha\beta$ is all-pass and

$$\left\|H_C\right\|_2 = \left\|\frac{\alpha\beta}{ab} H_C\right\|_2 = \left\|\frac{\alpha y \beta}{a} - \alpha W \beta\right\|_2$$

H_2 Optimal Control

Consider the decomposition

$$\frac{\alpha y \beta}{a} = r + \frac{q}{a}$$

with r polynomial and q/a strictly proper.

With this decomposition,

$$\|H_C\|_2^2 = \left\|\frac{q}{a}\right\|_2^2 + \|r - \alpha W \beta\|_2^2$$

because q/a and $r - \alpha W \beta$ are orthogonal

and thus the cross-terms contribute nothing to the norm.

The last expression is a complete square

whose first term is independent of W.

Hence the minimum is *unique* and achieved for $W = r/\alpha\beta$.

H_2 Optimal Control

The H_2 optimal control is a special case of pole placement. Indeed,

$$R = \frac{y - a\dfrac{r}{\alpha\beta}}{x + b\dfrac{r}{\alpha\beta}} = \frac{\alpha y\beta - ar}{\alpha x\beta + br} := \frac{q}{p}$$

and

$$ap + bq = a(\alpha x\beta + br) + b(\alpha y\beta - ar) = \alpha\beta(ax + by) = \alpha\beta$$

The optimal closed-loop poles are given by $\alpha\beta$.

Example 6

Plant

$$S(s) = \frac{1}{s-1}$$

Stabilizing controllers

$$R(s) = \frac{1-(s-1)W}{W}, \ W \neq 0 \text{ stable rational}$$

The complementary sensitivity function to be minimized is

$$H_C(s) = 1-(s-1)W$$

Now $\alpha = s + 1, \beta = 1$

and the polynomial part of $\alpha\gamma\beta/a = (s+1)/(s-1)$ is $r = 1$.

Thus H_C attains minimum H_2 norm for

$$W = \frac{1}{s+1}$$

and the corresponding optimal controller is $R(s) = 2$.

Example 6

Alternatively, one can solve the Diophantine equation

$$(s-1)p + q = s + 1$$

for the solution pair *p*, *q* such that *q*/(*s* –*1*) is strictly proper. This yields the least-degree solution pair with respect to *q*, namely *p* = 1, *q* = 2.

The optimal controller is *R*(*s*) = *q*/*p* = 2.

In general, it is simpler to solve the polynomial equation than calculating with rational functions.

l_1 Optimal Control

The H_2 norm minimization is appropriate
for systems excited by finite energy signals.
When the exogenous signals persist,
a more relevant norm to measure system performance
is the L_1 norm (for continuous-time systems)
or the l_1 norm (for discrete-time systems).
The discrete-time case is much easier.

Plant $S = b/a$
Find a stabilizing controller $R = Y/X$
such that, say, $H_S = a(x + bW)$ has a least l_1 norm.

l_1 Optimal Control

The optimal sensitivity function $H_S = a(x + bW)$ is not unique but it has a FIR property.

Perform stable-unstable factorizations $a = a^+a^-$ and $b = b^+b^-$, where a^- and b^- absorb all the zeros of a and b, respectively, in the open unit disc $|z^{-1}| < 1$.

Then H_S is a polynomial in z^{-1} iff W has the form

$$W = \frac{w}{a^+b^+},$$

where w is a free polynomial.

Indeed, $H_S = ax + a^-b^-w$

and the l_1-norm minimization of H_S is equivalent to a finite linear program for the coefficients of w.

Example 7

Plant

$$S(z^{-1}) = z^{-1}\frac{z^{-1}-1.5}{(1-z^{-1})^2}$$

Equation

$$(1-z^{-1})^2 x + z^{-1}(z^{-1}-1.5)y = 1$$

A solution $x = 1 - 0.5z^{-1}$, $y = -3 + 2z^{-1}$

yields the set of stabilizing controllers

$$R(s) = \frac{-3+2z^{-1}-(1-2z^{-1})^2 W}{1-0.5z^{-1}+z^{-1}(z^{-1}-1.5)W}$$

for any stable rational parameter *W*.

Example 7

The set of achievable sensitivity functions is

$$H_S(z^{-1}) = (1-2z^{-1})^2(1-0.5z^{-1}) + z^{-1}(1-2z^{-1})^2(z^{-1}-1.5)W$$

and those which are *polynomials* in z^{-1} are

$$H_S(z^{-1}) = (1-2z^{-1})^2(1-0.5z^{-1}) + z^{-1}(1-2z^{-1})^2 w$$

where w is the numerator polynomial in z^{-1} of

$$W = \frac{w}{z^{-1}-1.5}$$

Example 7

An upper bound for the degree of w is 2. The linear program:

minimize $t = r_1 + r_2 + r_3 + r_4 + r_5$

subject to $-r_i \le h_i \le r_i$ and $r_i \ge 0$, $i = 1, 2, \ldots, 5$

where

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{bmatrix} = \begin{bmatrix} -4.5 \\ 6 \\ -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 4 & -4 & 1 \\ 0 & 4 & -4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}$$

then returns $w_0 = 1.5$, $w_1 = 0$, $w_2 = 0$ so that

$$W = \frac{1.5}{z^{-1} - 1.5}$$

Example 7

The optimal controller is

$$R(s) = \frac{3-4z^{-1}}{(1+z^{-1})(z^{-1}-1.5)},$$

the corresponding optimal sensitivity function is

$$H_S(s) = 1-3z^{-1}+4z^{-2}.$$

It is to be noted that *R* is *not* a deadbeat controller because SH_S is not a polynomial.
Indeed, only polynomial parameters *W* result in deadbeat controllers.

Robust Stabilization

The notion of robust stability
addresses stabilization of plants subject to modeling errors,
when the actual plant may differ from the nominal model,
using a *fixed* controller.

The ultimate goal is to stabilize the actual plant.
The actual plant is unknown, however,
so the best one can do is to stabilize a large enough set of plants.

The set of plants is constructed
as a neighborhood of the nominal plant.
The size of the neighborhood is measured by a suitable norm,
most common being the H_∞ norm.

Model of Uncertainty

Consider a nominal plant with transfer function S
and its neighborhood S_Δ defined by $S_\Delta := (1 + \Delta F)S$,
where F is a *fixed* stable rational function
and Δ is a *variable* stable rational function such that $\|\Delta\|_\infty \leq 1$.
Note that ΔF is the normalized plant perturbation away from 1

$$\frac{S_\Delta}{S} - 1 = \Delta F$$

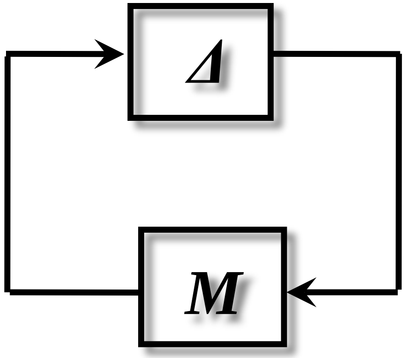
Hence if $\|\Delta\|_\infty \leq 1$, then for all frequencies ω

$$\left|\frac{S_\Delta(j\omega)}{S(j\omega} - 1\right| \leq |F(j\omega)|$$

so $|F(j\omega)|$ provides the uncertainty profile
while Δ accounts for phase uncertainty.

Small Gain Theorem

Consider the M-Δ feedback system:



Suppose that M is stable.

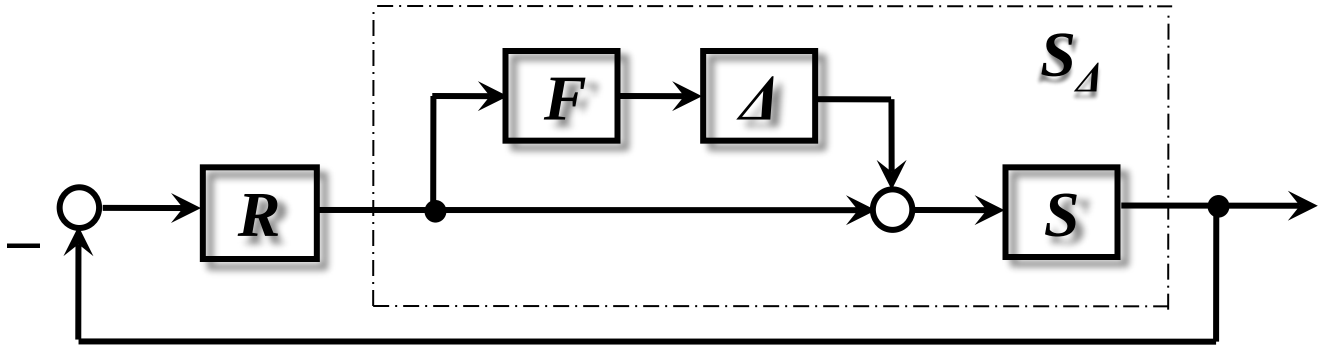
Then the feedback system is stable

for all stable Δ with $\|\Delta\|_\infty \leq 1$

if and only if $\|M\|_\infty < 1$.

Robust Stability Condition

The given model of uncertainty



collapses to an M**-**Δ **feedback system with** $M = -F\dfrac{SR}{1+SR}$

Suppose that R **stabilizes the nominal plant** S**.**

Then R **will stabilize the entire family of plants** S_Δ **iff**

$$\left\| FH_C \right\|_\infty < 1$$

Parameterized Condition

The set of all stabilizing controllers for $S = b/a$ is described by the formula

$$R = -\frac{y - aW}{x + bW}$$

where $ax + by = 1$ and W is a free stable rational parameter. The robust stability condition then reads

$$\left\| Fb(y - aW) \right\|_{\infty} < 1$$

Any stable rational W that satisfies this inequality then defines a robustly stabilizing controller R for S. In case W actually minimizes the norm one obtains the best robustly stabilizing controller.

Example 8

Plant

$$S_{\tau}(s) = \frac{s+1}{s-1} e^{-\tau s}$$

where the time delay τ is known only to the extent that it lies in the interval $0 \le \tau \le 0.2$.

Find a controller that stabilizes the uncertain plant S_{τ}.

Example 8

The time-delay factor can be treated as a multiplicative perturbation of the nominal plant

$$S(s) = \frac{s+1}{s-1}$$

by embedding S_τ in the family $S_\Delta := (1 + \Delta F)S$, where Δ ranges over the set of stable rational functions such that $\|\Delta\|_\infty \le 1$.

Example 8

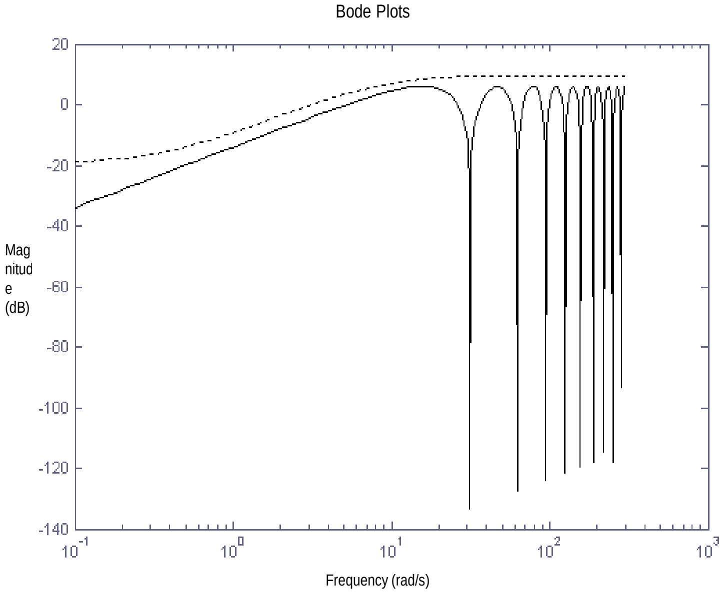
To do this, *F* should be chosen so that

$$\left|\frac{S_{\Delta}(j\omega)}{S(j\omega)}-1\right| = \left|e^{-j\omega\tau}-1\right| \leq \left|F(j\omega)\right|$$

A suitable uncertainty profile is

$$F(s) = \frac{3s+1}{s+9}$$

Bode magnitude plot of this *F* and $e^{j\omega\tau}-1$ for $\tau = 0.2$, the worst value



Example 8

The set of all stabilizing controllers

for the nominal plant *S* is

$$R(s) = \frac{\frac{1}{2} - (s-1)W}{-\frac{1}{2} + (s+1)W}$$

where $W \neq 1/2(s + 1)$ is any stable rational parameter.

Example 8

The robust stability condition reads

$$\left\| P - QW \right\|_\infty < 1$$

where

$$P(s) = \tfrac{1}{2}(s+1)\frac{3s+1}{s+9}, \;\; Q(s) = (s-1)(s+1)\frac{3s+1}{s+9}$$

The maximum modulus theorem implies
that the minimum of the H_∞ norm
taken over all stable rational functions W is achieved for

$$W(s) = \frac{P(s) - P(1)}{Q(s)} = \frac{1}{10}\frac{15s+31}{(s+1)(3s+1)}$$

Example 8

Thus the robust stability condition is satisfied and the best robustly stabilizing controller is

$$R(s) = \frac{2}{13}\frac{s+9}{s+1}$$

Stabilization
Subject to Input Constraints

Most plants have inputs that are subject to hard limits on the range of variations that can be achieved.

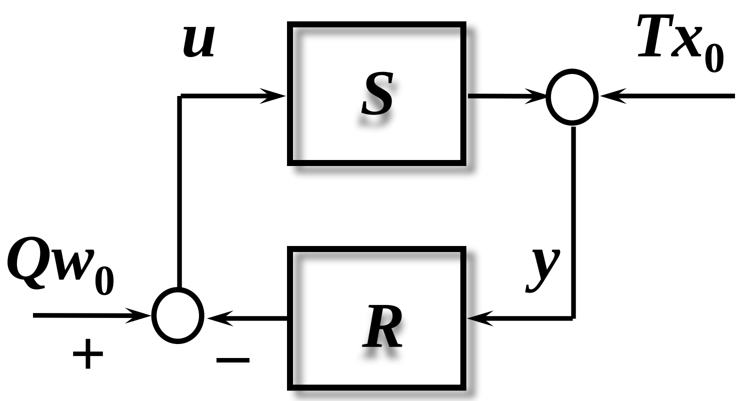
Stabilization subject to input constraints:

local stabilization,
 saturation prevented for a set of initial states,
 the control system behaves as a linear one

global stabilization,
 saturation occurs, the control system is nonlinear

Problem Formulation

Discrete-time control system



Find

a controller *R* such that

the control system is *locally asymptotically stable*

for any initial state $x_0 \in P_F$

$$P_F = \{\, x: Fx \le f \,\}$$ **polyhedron**

and $u(z) = u_0 + u_1 z^{-1} + u_2 z^{-2} + \ldots$

$$-u^- \le u_k \le u^+$$ **constraint**

Controller Parameterization

Stabilizing controllers $R = Y/X$

$$X = x + bW, \quad Y = y - aW$$

Control sequence ($w_0 = 0$ **assumed)**

$$u = -c\,(y - aW)\,x_0\,, \qquad W = p_0 + p_1 z^{-1} + \ldots$$

is a linear function of the parameters $p_0, p_1, \ldots$ **of the form**

$$u_k = G_k(p_1, p_2, \ldots),\ k = 0, 1, \ldots$$

and it satisfies the constraint if x_0 **is in** $P_G = \{\, x\colon G(w)\,x \le g \,\}$
where

$$G(p_0, p_1, \ldots) = \begin{bmatrix} G_0(w) \\ -G_0(w) \\ G_1(w) \\ -G_1(w) \\ \vdots \end{bmatrix}, \qquad g = \begin{bmatrix} u^+ \\ u^- \\ u^+ \\ u^- \\ \vdots \end{bmatrix}$$

Polyhedron Inclusion

Now x_0 is in P_F, so P_F must be contained in P_G.

***Farkas lemma*:**

A polyhedron $P_F = \{ x: Fx \le f \}$
is contained
in a polyhedron $P_G = \{ x: Gx \le g \}$
if and only if there exists a matrix P
with non-negative entries
such that

$$PF = G, \quad Pf \le g$$

Solution

The problem has a solution
if and only if there exist a matrix *P* with non-negative entries and real numbers p_0, p_1, … such that

$$PF = G(p_0, p_1, \ldots), \quad Pf \leq g$$

This is a *linear program* for *P* and p_0, p_1, …

The stabilizing controller is then obtained by putting

$$W = p_0 + p_1 z^{-1} + \ldots$$

The program has a finite dimension
if *W* is approximated by a polynomial.

Example 9

Consider a plant described by input-output and state-output transfer functions of the form

$$S(z) = \frac{z^{-1}}{1 - z^{-1}}, \quad T(z) = \frac{2}{1 - z^{-1}}$$

The corresponding state equation

$$x_{k+1} = x_k + 0.5u_k, \quad y_k = 2x_k$$

The plant input is constrained as

$$-1 \le u_k \le 1, \quad k = 0,1,\ldots$$

and the initial state x_0 belongs to the polyhedron

$$P_F : \begin{bmatrix} 1 \\ -1 \end{bmatrix} x_0 \le \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \quad (\text{or } |x_0| \le 1/3).$$

Example 9

Stabilizing controllers

$$R(z) = \frac{2-(1-2z^{-1})W}{1+z^{-1}W}$$

for a free, proper stable rational parameter *W*.
The corresponding control sequence is

$$u(z) = \left[-4+2(1-2z^{-1})W\right]x_0$$

Now start with *W* = 0 and check whether the resulting linear program for *P* is feasible:

$$P\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-4\\4\end{bmatrix}, \quad P\begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\end{bmatrix} \leq \begin{bmatrix}1\\1\end{bmatrix}$$

It is not, hence no controller of order 0 stabilizes the plant.

Example 9

Proceed by choosing $W = p_0$ and check whether the resulting linear program for p_0 and P is feasible:

$$P\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4+2p_0 \\ 4-2p_0 \\ -4p_0 \\ 4p_0 \end{bmatrix}, \quad P\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

It is, and the solution

$$p_0 = \frac{2}{3}, \quad P = \begin{bmatrix} 0 & 8 \\ 8 & 0 \\ 0 & 8 \\ 8 & 0 \end{bmatrix}$$

furnishes a stabilizing controller

$$R(z) = \frac{4+4z^{-1}}{3+2z^{-1}}$$

Example 9

The actual polyhedron of stabilizable initial states is

$$P_G: \frac{1}{3}\begin{bmatrix} -8 \\ 8 \\ -8 \\ 8 \end{bmatrix} x_0 \le \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{(or } |x_0| \le 3/8\text{)}$$

and it includes P_F as a proper subset.

Note that the closed-loop control system features the finite impulse response property.

Selecting a polynomial parameter W implies that the closed-loop poles are all at the origin.

Input and Output Shaping

Input constraints, but also output overshoot or undershoot

In discrete time, easy to handle.

The *z*-transform provides a simple direct relationship

$$(y_0, y_1, y_2, \ldots) \leftrightarrow y_0 + y_1 z^{-1} + y_2 z^{-2} + \ldots$$

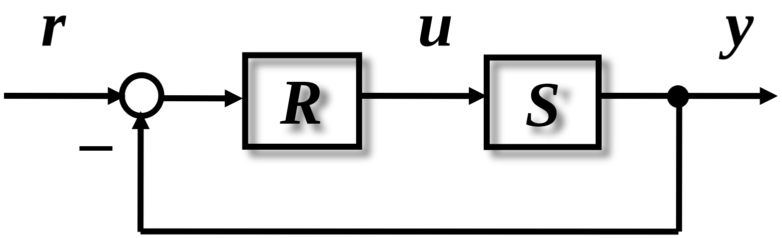
Time domain constraints boil down to constraints on polynomial coefficients.

In continuous time, a new approach is needed:

- **assign distinct negative real poles (rather than poles at $z = 0$)**
- **express time signals as polynomials in the corresponding exponential modes**

Problem Formulation

Given a plant $S = b/a$,
we are seeking a stabilizing controller $R = q/p$
such that the output y asymptotically follows a reference r



while the time-domain constraints
$u_{\min} \le u(t) \le u_{\max}, \quad y_{\min} \le y(t) \le y_{\max}$ **are satisfied for all $t \ge 0$,**
where $u_{\min}$, $u_{\max}$, $y_{\min}$, and $y_{\max}$ are given real numbers.

We assume that S is strictly proper
and that R is proper so as to avoid impulsive modes.

Pole Assignment

Assign distinct negative *integer* poles

$$ap + bq = d \;:=\; \prod_i (s - s_i)$$

Then signals are sums of decaying exponentials modes

$$u(t) = \sum_i u_i e^{-s_i t}, \quad y(t) = \sum_i y_i e^{-s_i t}$$

Let g be the greatest common divisor of the poles s_i

so that $s_i = k_i g$ for some integers k_i.

The signals can now be expressed as polynomials in $\lambda = e^{-gt}$

$$u(\lambda) = \sum_i u_i \lambda^{k_i}, \quad y(\lambda) = \sum_i y_i \lambda^{k_i}$$

Polynomial Non-Negativity Constraints

When time t increases from 0 to ∞,

indeterminate λ decreases from 1 to 0

and the time constraints become the polynomial constraints

$$u_{\min} \le u(\lambda) \le u_{\max}, \quad y_{\min} \le y(\lambda) \le y_{\max}$$

or, equivalently, the polynomial non-negativity constraints

$$u(\lambda) - u_{\min} \ge 0, \quad -u(\lambda) + u_{\max} \ge 0,$$
$$y(\lambda) - y_{\min} \ge 0, \quad -y(\lambda) + y_{\max} \ge 0,$$

along the interval λ ∈ [0, 1].

Convex LMI Constraint

A polynomial non-negativity constraint

$$p(\lambda) = \sum_{i=0}^{n} p_i \lambda^i \geq 0, \quad \text{for all } \lambda \in [\lambda_{\min}, \lambda_{\max}]$$

is equivalent to the existence
of real symmetric matrices $P_{\min}$, $P_{\max}$ of size $n + 1$
satisfying the linear matrix inequality constraints

$$p_i = \text{trace}[\mathrm{P}_{\min}(H_{i-1} - \lambda_{\min} H_i) + P_{\max}(\lambda_{\max} H_i - H_{i-1})], \quad i = 0, 1, \ldots, n$$

$$P_{\min} \geq 0, \quad P_{\max} \geq 0$$

where H_i is the basis Hankel matrix
with ones along the (i + 1)th anti-diagonal
and zeros elsewhere.

Design Parameters

Now all proper rational controllers *R*
that assign the pole polynomial $d := \Pi_i (s - s_i)$
are parameterized by a numerator polynomial *w*
of appropriate degree.

The coefficients of *w* are our design parameters
and they appear in the coefficients u_i, y_i
of the closed-loop signals in an affine manner.

Therefore the linear matrix inequalities are convex
in the design parameters.

Example 10

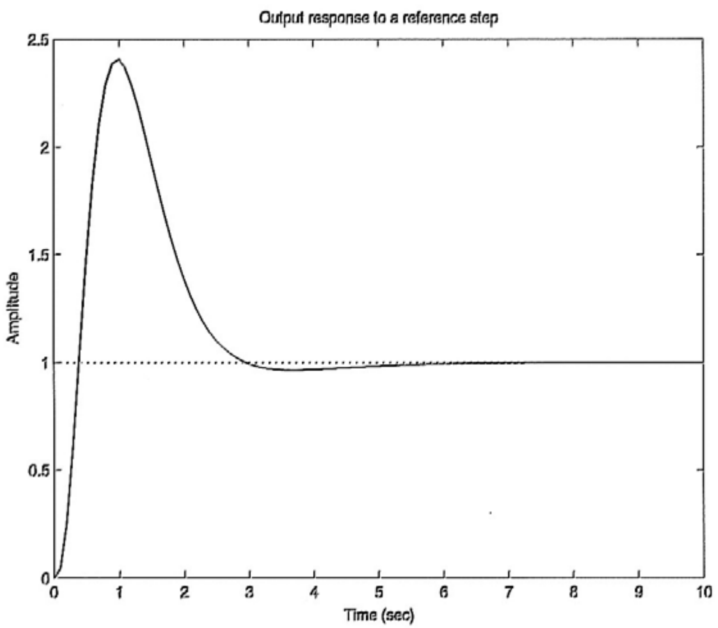
Given the plant

$$S(s) = \frac{s+0.5}{s(s-2)}$$

the stabilizing controller

$$R(s) = \frac{384s+240}{s^3+17s^2+119s+79}$$

assigns the closed-loop poles at – 1, – 2, – 3, – 4, – 5 while ensuring asymptotic step reference tracking.



Despite the poles being negative real, the step response features an unacceptable overshoot of 140 % due to system zeros.

Example 10

The set of all proper rational controllers that assign the above poles is given by

$$R(s) = \frac{384s + 240 - s(s-2)w}{s^3 + 17s^2 + 119s + 79 + (s+0.5)w}$$

where $w = w_0 + w_1 s$ is a free polynomial of degree at most 1.

Example 10

The closed-loop responses to a step input are affine in *w*,

$$y(s) = \frac{384s^2 + 432s + 120 - (s^3 - 1.5s^2 - s)w}{(s+1)(s+2)(s+3)(s+4)(s+5)}$$

and correspond to a sum of decaying exponential modes in the time domain,

$$y(t) = \sum_{i=0}^{5} y_i e^{-it}$$

or to a polynomial

$$y(\lambda) = \sum_{i=0}^{5} y_i \lambda^i$$

in the indeterminate $\lambda = e^{-t}$ **.**

The coefficients y_i **are *linear* functions of** w_0 **and** w_1**.**

Example 10

Suppose the desired maximum overshoot is 20%

$$y(t) \leq 1.2\, y_0$$

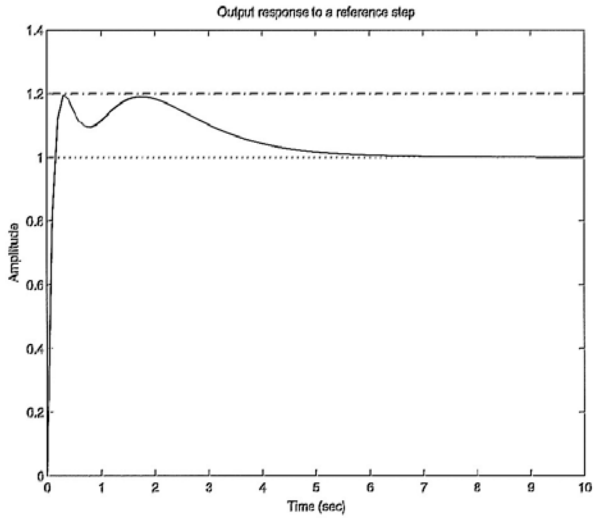
equivalent to the polynomial non-negativity constraint

$$p(\lambda) = 1.2 y_0 - y(\lambda) = 0.2 y_0 - y_1\lambda - y_2\lambda^2 - y_3\lambda^3 - y_4\lambda^4 - y_5\lambda^5 \geq 0$$

and in turn equivalent to an LMI in w_0 and w_1. The LMI returns

$$w(s) = -100.36 - 12.27s$$

keeping the controller of order 3.



Fixed-Order Stabilizing Controllers

A weakness of the sequential design based on the Youla-Kučera parameterization is that each performance specification beyond stability may *increase the order* of the controller.

Actually, fixed-order stabilizing controllers can be found by solving an LMI.

Polynomial Degree Control

The degree control in the parameter $W = w/d$ is difficult.

If d is fixed, all closed-loop transfer functions are affine in w but the order of w increases with each additional specification. If d is not fixed, we have a greater flexibility but we run into difficulties as the set of stable polynomials is not convex in the space of coefficients.

The difficulty was resolved by providing a *convex inner approximation* of the non-convex stability domain in the space of polynomial coefficients. This approximation is parameterized by a given polynomial, referred to as the *central* polynomial.

Problem Formulation

Let us now show how to design stabilizing controllers of a fixed (presumably low) order.

Suppose a plant $S = b/a$ is given and suppose that we have a stabilizing controller $\bar{R} = q/p$. We seek to find a stabilizing controller $R = y/x$ of a given order m, if such a controller exists.

The Two Controllers Relationship

The two stabilizing controllers are related as

$$p = x + bW, \quad q = y - aW, \quad \text{where } W = w/d.$$

Then

$$\underbrace{\begin{bmatrix} d & 0 & -p & b \\ 0 & d & -q & -a \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ d \\ w \end{bmatrix} = 0.$$

Minimal Polynomial Basis

Let

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ d_1 & d_2 \\ w_1 & w_2 \end{bmatrix}$$

be a minimal polynomial basis of *A*.

Then all stabilizing controllers for *S* are

$$R = (\lambda_1 y_1 + \lambda_2 y_2)/(\lambda_1 x_1 + \lambda_2 x_2)$$

where λ_1 and λ_2 are polynomials

such that $\lambda_1 d_1 + \lambda_2 d_2$ is a *stable* polynomial.

A stabilizing controller of order *m* exists if

$$\deg \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = m$$

Alas, the set of *stable* polynomials is not convex.

Linear Matrix Inequality

Given a fixed stable "central" polynomial *c*(*s*) of degree *n*, polynomial *d*(*s*) of degree *n* is stable if there exists a real symmetric matrix *Q* of size *n* solving the linear matrix inequality

$$H_c(d,Q) = c^T d + d^T c - \varepsilon c^T c + \Pi_1^T Q \Pi_2 + \Pi_2^T Q \Pi_1 \geq 0$$

where

$$\Pi_1 = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \end{bmatrix}$$

are projection matrices,
***c* and *d* are the coefficient vectors of *c*(*s*) and *d*(*s*),**
and ε is an arbitrarily small positive scalar.

Convex Inner Approximation

The interpretation of this result is as follows:

as soon as polynomial *c* is fixed,

we obtain a sufficient linear matrix inequality condition

for stability of polynomial *d*.

Therefore,

$$H_c = \{d : \exists Q : H_c(d, Q) \geq 0\}$$

is a convex inner approximation of the (generally non-convex) stability domain in the space of polynomial coefficients around the central stable polynomial.

Problem Solution

Using the convex inner approximation
of the set of stable polynomials,

we can optimize over polynomials λ_1 and λ_2
to enforce low degrees of x and y (linear algebraic constraint)

$$\deg \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = m$$

as well as stability of d (linear matrix inequality constraint)

$$\begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = d$$

Example 11

Consider a plant of order 3,

$$S(s) = \frac{1}{s(s^2 + s + 10)}.$$

A stabilizing controller of order 2
can be found by placing the closed-loop poles at arbitrary locations. For example, the controller

$$\overline{R}(s) = \frac{-26s^2 + 45s + 1}{s^2 + 4s - 4}$$

places all five closed-loop poles at -1.

Find a lower order stabilizing controller.

Example 11

A minimal polynomial basis for the polynomial matrix A is

$$\begin{bmatrix} 0 & 1 \\ -1 & -26 \\ -1 & s^3+s^2+10s-26 \\ s^2+4s-4 & 149s-103 \end{bmatrix}$$

All the stabilizing controllers

can be recovered from the polynomials λ_1 and λ_2

such that the pole polynomial

$$d = -\lambda_1 + \lambda_2(s^3 + s^2 + 10s - 26)$$

is stable.

Example 11

From the first two rows of the basis a controller of order 0 can be obtained by restricting the parameters λ_1 and λ_2 to be constant.

Hurwitz stability criterion then reveals that d is stable if and only if $\lambda_1 \in (-36, -26)$ and $\lambda_2 = 1$.

For example, with $\lambda_1 = -30$ we obtain the controller $R(s) = 4$ and the closed-loop pole polynomial $d(s) = s^3 + s^2 + 10s + 4$.

In this example, we were able to obtain an exact solution.
In general, the linear matrix inequality has to be used.

Summary

The benefits of representing stabilizing controllers by a single parameter

- **easy accommodation of additional design specifications by selecting an appropriate parameter**
- **all transfer functions in a stabilized system are *linear* in the parameter (while they are nonlinear in the controller)**
- **the parameter belongs to a smaller set of *stable* rational functions (while the controller is any rational)**

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INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

State Space Representation of All Stabilizing Controllers

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Tato prezentace je spolufinancována Evropským sociálním fondem a státním rozpočtem České republiky.

The Importance of Stabilization

Most control systems are required to be stable
and to meet additional performance specifications,
such as optimality or robustness.

It is natural to design the systems step by step:
stabilization first,
then the additional specifications each at a time.

For this it is obviously necessary to have
any and all solutions of the current step available
before proceeding any further.

The Importance of Parameterization

This motivates the need for all controllers that stabilize a given system.

This is an infinite set and we find it convenient to describe it in a parametric form, known as the Youla-Kučera parameterization.

The additional specifications are then met by selecting an appropriate parameter.

Such a procedure is simple, systematic, and transparent.

Definition of Stability

A (linear, time invariant, differential) system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

is said to be (asymptotically) *stable*
if any initial state $x(0)$ goes to zero as $t \to \infty$.

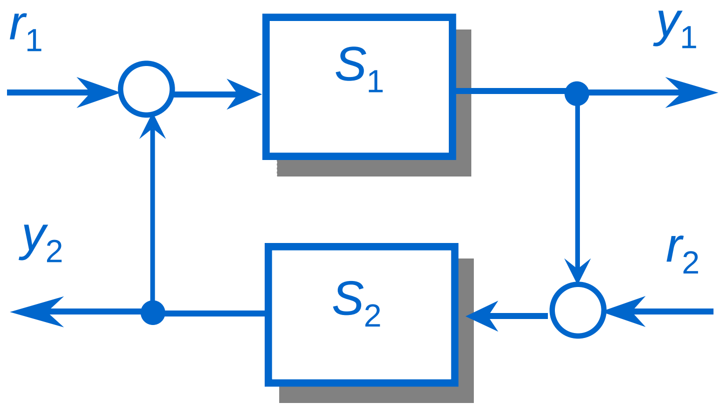
A system is stable
iff all eigenvalues of A have negative real part.

A system that is *controllable and observable* is stable
iff all poles of the transfer function $H(s) = C(sI - A)^{-1}B + D$
have negative real part.

Feedback System

Stabilization is achieved by feedback.

The generic form of a feedback system



The system with inputs r_1, r_2 and outputs y_1, y_2
is *controllable and observable*
whenever the constituent systems S_1 and S_2 are so.

Transfer Function

Transfer function $H(s)$ that relates r_1, r_2 and y_1, y_2

$$H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} I & -H_2 \\ -H_1 & I \end{bmatrix}^{-1} = \begin{bmatrix} -H_2 & I \\ I & -H_1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & H_2 \\ H_1 & 0 \end{bmatrix}$$

where

$$H_1(s) = C(sI - A)^{-1}B + D, \quad H_2(s) = \overline{C}(sI - \overline{A})^{-1}\overline{B} + \overline{D}$$

are the (proper rational) transfer functions of S_1 and S_2.

The controllable and observable feedback system is stable iff $H(s)$ is a *proper stable* rational matrix (all poles within the open left half-plane).

A Special Case: SISO Systems

Write

$$H_1(s) = \frac{N(s)}{D(s)}$$

where *D*, *N* are *proper stable rational functions* rather than polynomials.

For example,

$$\frac{1}{s-1} = \left(\frac{1}{s+a}\right)\left(\frac{s-1}{s+a}\right)^{-1} = \left(\frac{s-1}{s+b}\right)^{-1}\left(\frac{1}{s+b}\right)$$

for any $a > 0$, $b > 0$.

An advantage of proper stable fractions:
H_1 is proper stable iff $1/D$ is proper stable.
The corresponding condition for *polynomial* fractions would involve a degree inequality.

A Special Case: SISO Systems

Write

$$H_1 = \frac{N}{D}$$

with D, N are coprime, *proper stable* rational functions.

Let X, Y be proper stable rational functions that satisfy the equation

$$DX + NY = 1.$$

Then *all proper rational* H_2 that stabilize the feedback system are given by

$$H_2 = -\frac{Y - DW}{X + NW}$$

where W is a proper stable rational parameter such that $1/(X + NW)$ exists and is proper.

MIMO Systems: Proper Stable Matrix Fractions

Write

$$H_1(s) = N_1(s)D_1^{-1}(s) = \tilde{D}_1^{-1}(s)\tilde{N}_1(s)$$

where D_1, N_1 and $\tilde{D}_1, \tilde{N}_1$ are *proper stable* rational matrices, with D_1, N_1 *right coprime* and $\tilde{D}_1, \tilde{N}_1$ *left coprime*.

For example,

$$H_1 = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s} \\ 0 & \frac{s+1}{s} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ 0 & s \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{s}{s+1} \end{bmatrix}^{-1} = N_1 D_1^{-1}$$

$$= \begin{bmatrix} s+1 & -1 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{s+1} \\ 0 & \frac{s}{s+1} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & 1 \end{bmatrix} = \tilde{D}_1 \tilde{N}_1$$

Parameterization of Stabilizing Controllers

Let

$$H_1 = N_1 D_1^{-1} = \tilde{D}_1^{-1} \tilde{N}_1$$

with D_1, N_1 right coprime and $\tilde{D}_1, \tilde{N}_1$ left coprime, proper stable matrix fractions.

Then *all proper rational* H_2
that stabilize the feedback system are given by

$$H_2 = -(X + W\tilde{N}_1)^{-1}(Y - W\tilde{D}_1) = -(\tilde{Y} - D_1\tilde{W})(\tilde{X} + N_1\tilde{W})^{-1}$$

where $W(s)$ and $\tilde{W}(s)$ are proper stable rational matrices such that the indicated inverses exist and are proper.

„Youla-Kučera parameterization“

Equivalence of Parameterizations

The two parameterizations of H_2 are equivalent.

To each controller H_2 there is a unique parameter W such that

$$H_2 = -(X + W\tilde{N}_1)^{-1}(Y - W\tilde{D}_1)$$

as well as a unique parameter $\tilde{W}$ such that

$$H_2 = -(\tilde{Y} - D_1\tilde{W})(\tilde{X} + N_1\tilde{W})^{-1}$$

and these two are related by

$$W - \tilde{W} = Y\tilde{X} - X\tilde{Y}.$$

Doubly Coprime Matrix Fractions

The proper stable matrix fractions

$$H_1 = N_1 D_1^{-1} = \tilde{D}_1^{-1} \tilde{N}_1$$

are said to be *doubly coprime*

if there exist proper stable rational matrices

X*, *Y* and $\tilde{X}, \tilde{Y}$ that satisfy the *Bézout identity

$$\begin{bmatrix} X & Y \\ -\tilde{N}_1 & \tilde{D}_1 \end{bmatrix} \begin{bmatrix} D_1 & -\tilde{Y} \\ N_1 & \tilde{X} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Doubly Coprime Matrix Fractions

Note that

$$\begin{bmatrix} X & Y \\ -\tilde{N}_1 & \tilde{D}_1 \end{bmatrix}\begin{bmatrix} D_1 & -\tilde{Y} \\ N_1 & \tilde{X} \end{bmatrix} =$$

$$\begin{bmatrix} XD_1 + YN_1 & Y\tilde{X} - X\tilde{Y} \\ \tilde{D}_1 N_1 - \tilde{N}_1 D_1 & \tilde{D}_1\tilde{X} + \tilde{N}_1\tilde{Y} \end{bmatrix} = \begin{bmatrix} I & W - \tilde{W} \\ 0 & I \end{bmatrix}$$

Thus doubly coprime matrix fractions provide a parameterization in which

$$W = \tilde{W}$$

and

$$H_2 = -X^{-1}Y = -\tilde{Y}\tilde{X}^{-1}$$

is a particular stabilizing controller, corresponding to $W = \tilde{W} = 0$.

Example

Given a plant with transfer function

$$H_1 = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s} \\ 0 & \frac{s+1}{s} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{s}{s+1} \end{bmatrix}^{-1} = N_1 D_1^{-1}$$

$$= \begin{bmatrix} 1 & -\frac{1}{s+1} \\ 0 & \frac{s}{s+1} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & 1 \end{bmatrix} = \tilde{D}_1 \tilde{N}_1$$

Determine the set of proper stabilizing controllers H_2 for H_1

Example

Solve the equation $XD_1 + YN_1 = I$

$$X\begin{bmatrix} 1 & 0 \\ 0 & \frac{s}{s+1} \end{bmatrix} + Y\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; Y = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

and the equation $\tilde{D}_1\tilde{X} + \tilde{N}_1\tilde{Y} = I$

$$\begin{bmatrix} 1 & -\frac{1}{s+1} \\ 0 & \frac{s}{s+1} \end{bmatrix}\tilde{X} + \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & 1 \end{bmatrix}\tilde{Y} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \tilde{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; \tilde{Y} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

Example

Then

$$H_2 = -(X + W\tilde{N}_1)^{-1}(Y - W\tilde{D}_1) = -(\tilde{Y} - D_1\tilde{W})(\tilde{X} + N_1\tilde{W})^{-1}$$

is given by

$$H_2(s) = -\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + W\begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1}\left(\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix} - W\begin{bmatrix} 1 & -\frac{1}{s+1} \\ 0 & \frac{s}{s+1} \end{bmatrix}\right)$$

$$= -\left(\begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{s+1} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & \frac{s}{s+1} \end{bmatrix}\tilde{W}\right)\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ 0 & 1 \end{bmatrix}\tilde{W}\right)^{-1}$$

where

$$W - \tilde{W} = Y\tilde{X} - X\tilde{Y} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Example

To have $W = \tilde{W}$, one replaces $\tilde{X}, \tilde{Y}$ with a different solution

$$\tilde{X} - N_1(W - \tilde{W}) = \begin{bmatrix} 1 & \frac{1}{s+1} \\ 0 & 1 \end{bmatrix}, \quad \tilde{Y} + D_1(W - \tilde{W}) = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{s+1} \end{bmatrix}$$

Then the Bézout identity holds

$$\begin{bmatrix} X & Y \\ -\tilde{N}_1 & \tilde{D}_1 \end{bmatrix} \underbrace{\begin{bmatrix} D_1 & -\tilde{Y} \\ N_1 & \tilde{X} \end{bmatrix} \begin{bmatrix} I & W - \tilde{W} \\ 0 & I \end{bmatrix}^{-1}}_{\begin{bmatrix} D_1 & -\tilde{Y} - D_1(W - \tilde{W}) \\ N_1 & \tilde{X} - N_1(W - \tilde{W}) \end{bmatrix}} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and

$$H_2 = -X^{-1}Y = -\tilde{Y}\tilde{X}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

is one stabilizing controller for H_1.

State Space Representations

Let $\dot{x} = Ax + Bu,\ y = Cx + Du$

be a *controllable and observable* realization of

$$H_1(s) = C(sI - A)^{-1}B + D := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Then doubly coprime,

proper stable matrix fractions for $H_1(s)$

can be obtained *directly* from the matrices A, B, C, D.

This will result in an alternative representation

of all proper rational H_2 that stabilize H_1

and provide useful insight.

Right Matrix Fraction for H_1

Consider a stabilizing state feedback $u = Fx + r$ around the system

$$\dot{x} = (A + BF)x + Br$$

$$u = Fx + r$$

$$y = (C + DF)x + Dr$$

Right Matrix Fraction for H_1

Consider a stabilizing state feedback $u = Fx + r$ around the system

$$\dot{x} = (A + BF)x + Br$$

$$u = Fx + r$$

$$y = (C + DF)x + Dr$$

Define

$$D_1(s) := \left[\begin{array}{c|c} A + BF & B \\ \hline F & I \end{array}\right],$$

Right Matrix Fraction for H_1

Consider a stabilizing state feedback $u = Fx + r$ **around the system**

$$\dot{x} = (A + BF)x + Br$$

$$u = Fx + r$$

$$y = (C + DF)x + Dr$$

Define

$$D_1(s) := \left[\begin{array}{c|c} A + BF & B \\ \hline F & I \end{array}\right], \quad N_1(s) := \left[\begin{array}{c|c} A + BF & B \\ \hline C + DF & D \end{array}\right]$$

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Define

$$D_1(s) := \left[\begin{array}{c|c} A+BF & B \\ \hline F & I \end{array}\right], \quad N_1(s) := \left[\begin{array}{c|c} A+BF & B \\ \hline C+DF & D \end{array}\right]$$

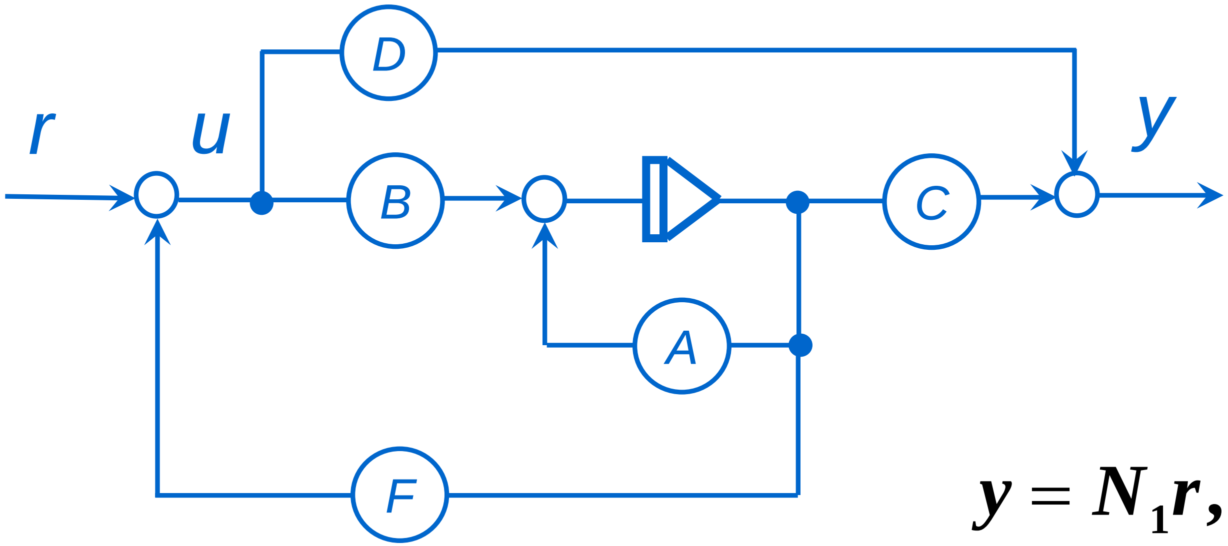
Then

$$y = N_1 r, \quad u = D_1 r$$

and

$$y = N_1 D_1^{-1} u = H_1 u$$

Right Matrix Fraction for H_1



$$y = N_1 r, \quad u = D_1 r$$

$$D_1(s) := \left[\begin{array}{c|c} A+BF & B \\ \hline F & I \end{array}\right], \quad N_1(s) := \left[\begin{array}{c|c} A+BF & B \\ \hline C+DF & D \end{array}\right]$$

Left Matrix Fraction for H_1

Consider a state observer for $\dot{x} = Ax + Bu,\ y = Cx + Du$ **based on a stabilizing output injection** Ke

$$\dot{\hat{x}} = (A - KC)\hat{x} + (B - KD)u + Ky$$

$$e = y - C\hat{x} - Du$$

Left Matrix Fraction for H_1

Consider a state observer for $\dot{x} = Ax + Bu,\ y = Cx + Du$ **based on a stabilizing output injection** Ke

$$\dot{\hat{x}} = (A - KC)\hat{x} + (B - KD)u + Ky$$

$$e = y - C\hat{x} - Du$$

Define

$$\tilde{D}_1(s) := \left[\begin{array}{c|c} A - KC & K \\ \hline -C & I \end{array}\right],$$

Left Matrix Fraction for H_1

Consider a state observer for $\dot{x} = Ax + Bu,\ y = Cx + Du$

based on a stabilizing output injection Ke

$$\dot{\hat{x}} = (A - KC)\hat{x} + (B - KD)u + Ky$$
$$e = y - C\hat{x} - Du$$

Define

$$\tilde{D}_1(s) := \left[\begin{array}{c|c} A - KC & K \\ \hline -C & I \end{array}\right], \quad \tilde{N}_1(s) := \left[\begin{array}{c|c} A - KC & B - KD \\ \hline C & D \end{array}\right]$$

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Consider a state observer for $\dot{x} = Ax + Bu,\ y = Cx + Du$

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$$e = y - C\hat{x} - Du$$

Define

$$\tilde{D}_1(s) := \left[\begin{array}{c|c} A - KC & K \\ \hline -C & I \end{array}\right], \quad \tilde{N}_1(s) := \left[\begin{array}{c|c} A - KC & B - KD \\ \hline C & D \end{array}\right]$$

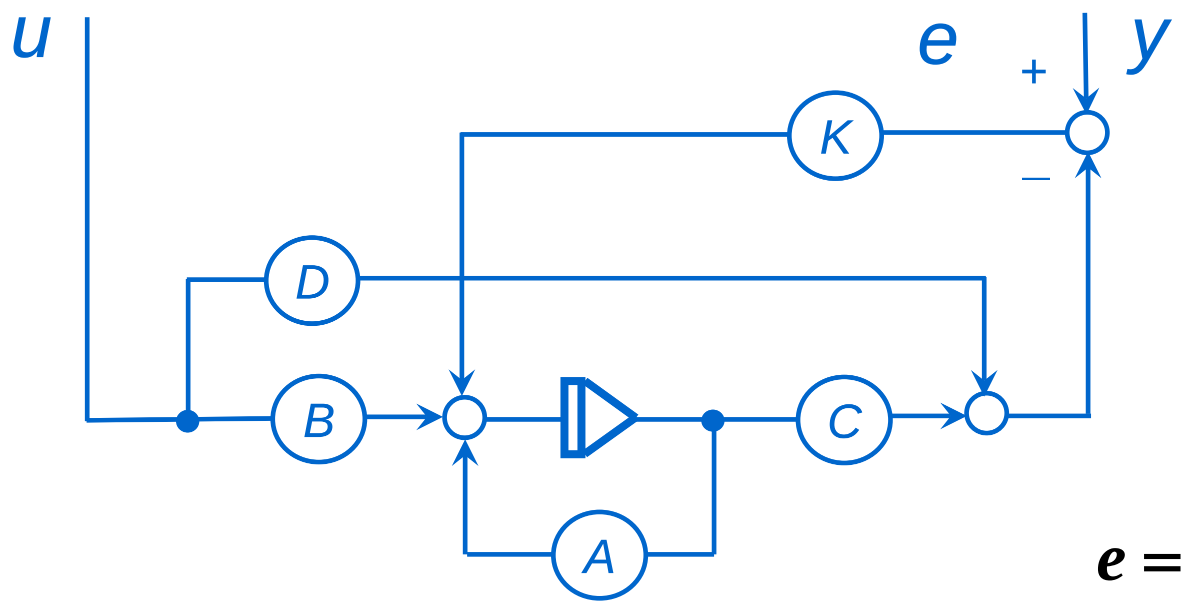
Then

$$e = \tilde{D}_1 y - \tilde{N}_1 u = (\tilde{D}_1 N_1 - \tilde{N}_1 D_1) r = 0$$

and

$$\tilde{D}_1^{-1}\tilde{N}_1 = N_1 D_1^{-1} = H_1$$

Left Matrix Fraction for H_1



$$e = \tilde{D}_1 y - \tilde{N}_1 u$$

$$\tilde{D}_1(s) := \left[\begin{array}{c|c} A-KC & K \\ \hline -C & I \end{array}\right], \quad \tilde{N}_1(s) := \left[\begin{array}{c|c} A-KC & B-KD \\ \hline C & D \end{array}\right]$$

Right Matrix Fraction for H_2

Consider a stabilizing state feedback around the observer with output *y* (*r* = 0) $u = F\hat{x}$

$$\dot{\hat{x}} = (A + BF)\hat{x} + Ke$$

$$u = F\hat{x}$$

$$y = (C + DF)\hat{x} + e$$

Right Matrix Fraction for H_2

Consider a stabilizing state feedback (*r* = 0) $u = F\hat{x}$
around the observer with output *y*

$$\dot{\hat{x}} = (A + BF)\hat{x} + Ke$$

$$u = F\hat{x}$$

$$y = (C + DF)\hat{x} + e$$

Define

$$\tilde{X}(s) := \left[\begin{array}{c|c} A + BF & K \\ \hline C + DF & I \end{array}\right],$$

Right Matrix Fraction for H_2

Consider a stabilizing state feedback around the observer with output y $(r = 0)$ $u = F\hat{x}$

$$\dot{\hat{x}} = (A + BF)\hat{x} + Ke$$
$$u = F\hat{x}$$

Define

$$y = (C + DF)\hat{x} + e$$

$$\tilde{X}(s) := \left[\begin{array}{c|c} A + BF & K \\ \hline C + DF & I \end{array}\right], \quad \tilde{Y}(s) := \left[\begin{array}{c|c} A + BF & K \\ \hline -F & 0 \end{array}\right]$$

Right Matrix Fraction for H_2

Consider a stabilizing state feedback (r = 0) $u = F\hat{x}$
around the observer with output *y*

$$\dot{\hat{x}} = (A + BF)\hat{x} + Ke$$
$$u = F\hat{x}$$
$$y = (C + DF)\hat{x} + e$$

Define

$$\tilde{X}(s) := \left[\begin{array}{c|c} A+BF & K \\ \hline C+DF & I \end{array}\right], \quad \tilde{Y}(s) := \left[\begin{array}{c|c} A+BF & K \\ \hline -F & 0 \end{array}\right]$$

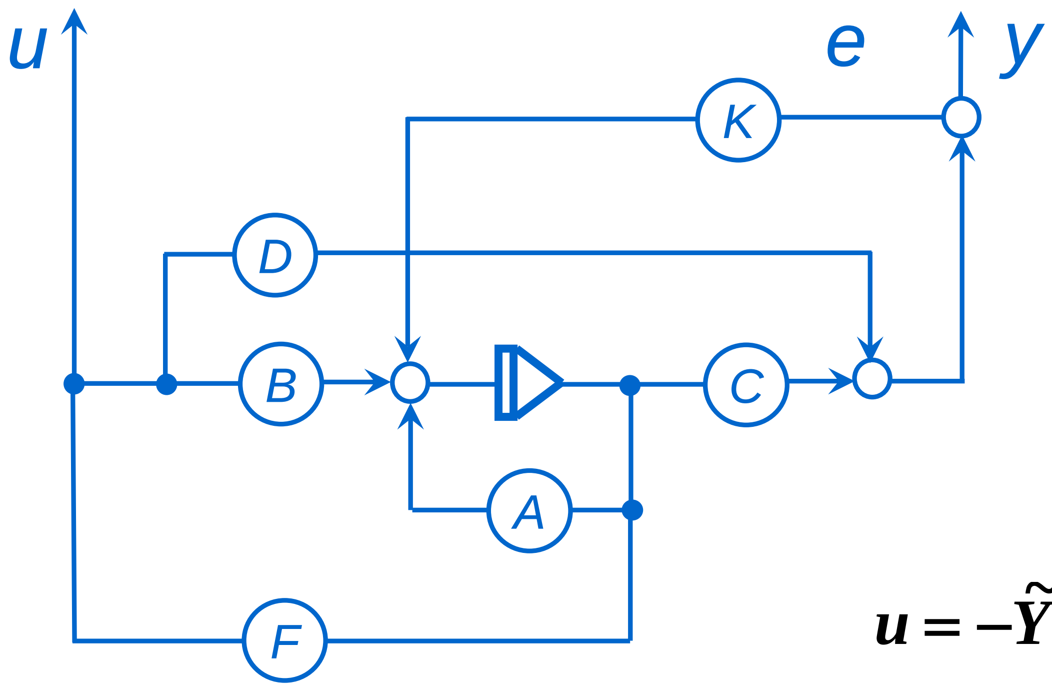
Then

$$u = -\tilde{Y}e, \quad y = \tilde{X}e$$

and

$$u = -\tilde{Y}\tilde{X}^{-1}y = H_2 y$$

Right Matrix Fraction for H_2



$$u = -\tilde{Y}e, \quad y = \tilde{X}e$$

$$\tilde{X}(s) := \left[\begin{array}{c|c} A+BF & K \\ \hline C+DF & I \end{array}\right], \quad \tilde{Y}(s) := \left[\begin{array}{c|c} A+BF & K \\ \hline -F & 0 \end{array}\right]$$

Left Matrix Fraction for H_2

Consider a stabilizing state feedback $u = F\hat{x} + r$
around the observer with output r

$$\dot{\hat{x}} = (A - KC)\hat{x} + (B - KD)u + Ky$$
$$r = -F\hat{x} + u$$

Left Matrix Fraction for H_2

Consider a stabilizing state feedback $u = F\hat{x} + r$ **around the observer with output** r

$$\dot{\hat{x}} = (A - KC)\hat{x} + (B - KD)u + Ky$$
$$r = -F\hat{x} + u$$

Define

$$X(s) := \left[\begin{array}{c|c} A - KC & B - KD \\ \hline -F & I \end{array}\right],$$

Left Matrix Fraction for H_2

Consider a stabilizing state feedback $u = F\hat{x} + r$ **around the observer with output** r

$$\dot{\hat{x}} = (A - KC)\hat{x} + (B - KD)u + Ky$$
$$r = -F\hat{x} + u$$

Define

$$X(s) := \left[\begin{array}{c|c} A - KC & B - KD \\ \hline -F & I \end{array}\right], \qquad Y(s) := \left[\begin{array}{c|c} A - KC & K \\ \hline -F & 0 \end{array}\right]$$

Left Matrix Fraction for H_2

Consider a stabilizing state feedback $u = F\hat{x} + r$ around the observer with output r

$$\dot{\hat{x}} = (A - KC)\hat{x} + (B - KD)u + Ky$$
$$r = -F\hat{x} + u$$

Define

$$X(s) := \left[\begin{array}{c|c} A - KC & B - KD \\ \hline -F & I \end{array}\right], \qquad Y(s) := \left[\begin{array}{c|c} A - KC & K \\ \hline -F & 0 \end{array}\right]$$

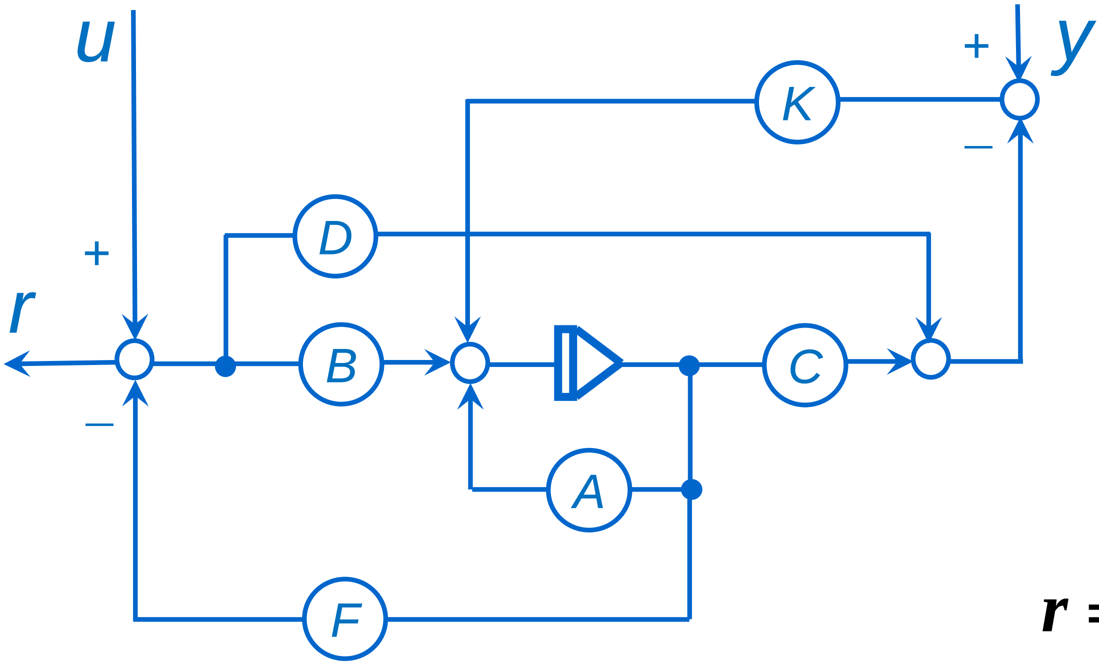
Then

$$r = Yy + Xu = (Y\tilde{X} - X\tilde{Y})e = 0$$

and

$$X^{-1}Y = \tilde{Y}\tilde{X}^{-1} = -H_2$$

Left Matrix Fraction for H_2



$$r = Yy + Xu$$

$$X(s) := \left[\begin{array}{c|c} A - KC & B - KD \\ \hline -F & I \end{array}\right], \quad Y(s) := \left[\begin{array}{c|c} A - KC & K \\ \hline -F & 0 \end{array}\right]$$

Doubly Coprime Fractions

Collecting the equations,

$$\begin{bmatrix} r \\ e \end{bmatrix} = \begin{bmatrix} X & Y \\ -\tilde{N}_1 & \tilde{D}_1 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}, \qquad \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} D_1 & -\tilde{Y} \\ N_1 & \tilde{X} \end{bmatrix} \begin{bmatrix} r \\ e \end{bmatrix}$$

the Bézout identity follows

$$\begin{bmatrix} X & Y \\ -\tilde{N}_1 & \tilde{D}_1 \end{bmatrix} \begin{bmatrix} D_1 & -\tilde{Y} \\ N_1 & \tilde{X} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Note that when $r = 0$,

$$H_2 = -X^{-1}Y = -\tilde{Y}\tilde{X}^{-1}$$

is a stabilizing controller for H_1.

Stabilizing Controllers Transfer Functions

Recall that

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} D_1 & -\tilde{Y} \\ N_1 & \tilde{X} \end{bmatrix} \begin{bmatrix} r \\ e \end{bmatrix}, \quad \begin{bmatrix} r \\ e \end{bmatrix} = \begin{bmatrix} X & Y \\ -\tilde{N}_1 & \tilde{D}_1 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}$$

Put $r = W(s)e$.

Then

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} D_1 & -\tilde{Y} \\ N_1 & \tilde{X} \end{bmatrix} \begin{bmatrix} r \\ e \end{bmatrix} = \begin{bmatrix} D_1 & -\tilde{Y} \\ N_1 & \tilde{X} \end{bmatrix} \begin{bmatrix} W \\ I \end{bmatrix} e = \begin{bmatrix} -(\tilde{Y} - D_1 W)e \\ (\tilde{X} + N_1 W)e \end{bmatrix}$$

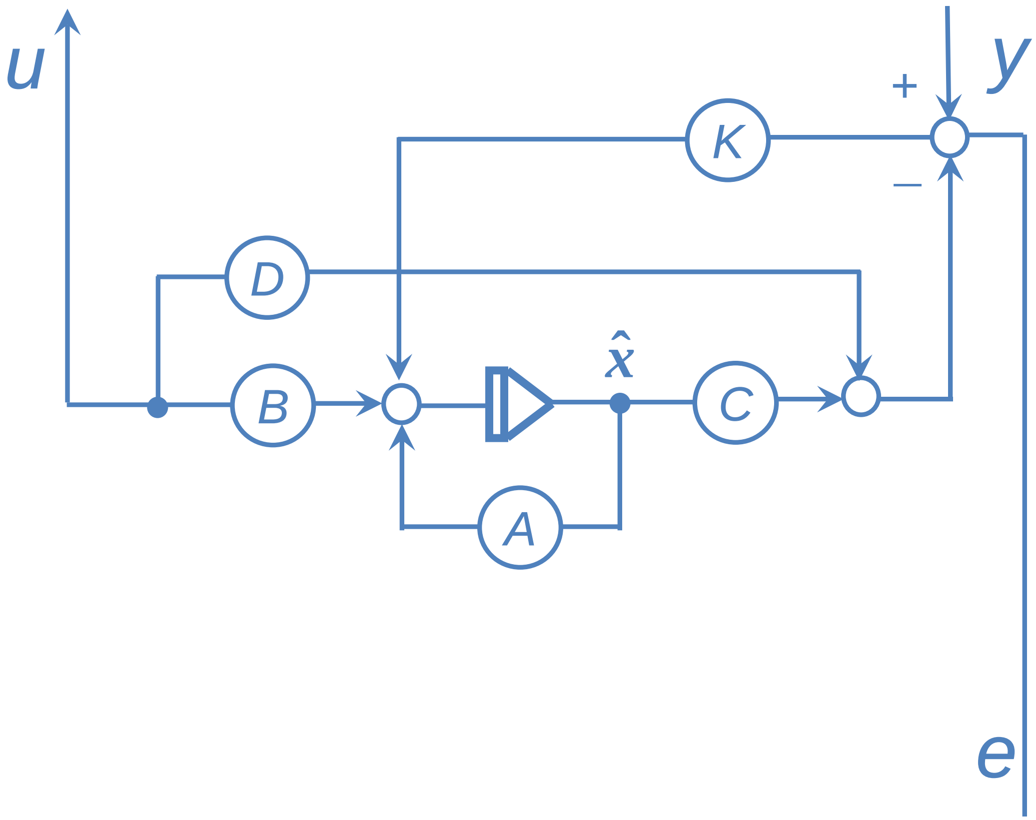
and

$$0 = [I \quad -W] \begin{bmatrix} r \\ e \end{bmatrix} = [I \quad -W] \begin{bmatrix} X & Y \\ -\tilde{N}_1 & \tilde{D}_1 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = (X + W\tilde{N}_1)u + (Y - W\tilde{D}_1)y$$

Hence

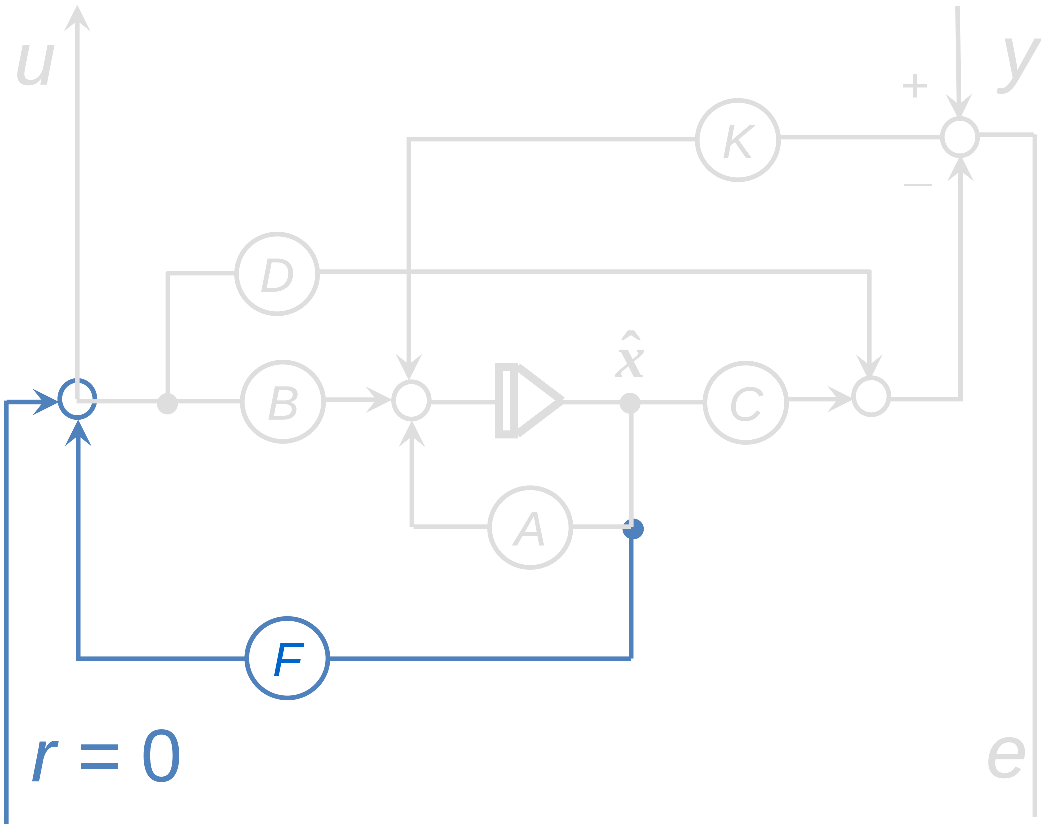
$$H_2 = -(X + W\tilde{N}_1)^{-1}(Y - W\tilde{D}_1) = -(\tilde{Y} - D_1 W)(\tilde{X} + N_1 W)^{-1}$$

Stabilizing Controllers Realization



observer

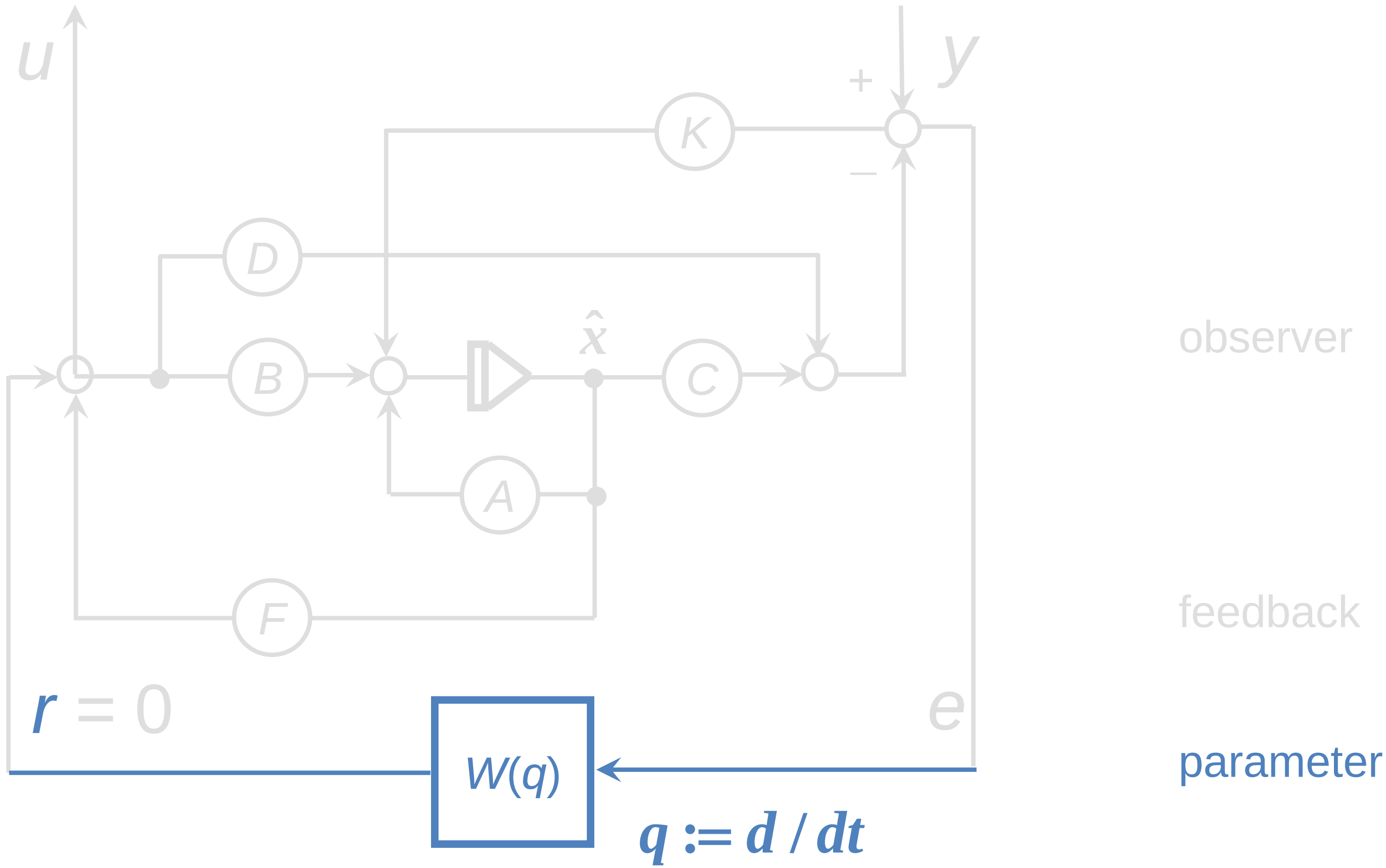
Stabilizing Controllers Realization



observer

feedback

Stabilizing Controllers Realization



Stabilizing Controllers Equations

All controllers that stabilize a given system are built around an observer-based controller (called *central* stabilizing controller)

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - (C\hat{x} + Du))$$

$$u = F\hat{x} + \underbrace{W(q)\underbrace{(y - (C\hat{x} + Du))}_{\text{output error } e}}_{\text{feedback correction } r}$$

Finding one stabilizing controller, we have them all.

Stabilizing Controllers Equations

central stabilizing controller, when $W = 0$

$$\dot{\hat{x}} = (A + BF - K(C + DF))\hat{x} + Ky$$
$$u = F\hat{x}$$

Stabilizing Controllers Equations

all stabilizing controllers, when $r = W(q)e$

$$\dot{\hat{x}} = (A + BF - K(C + DF))\hat{x} + Ky + (B - KD)r$$

$$u = F\hat{x} + r$$

$$r = W(q)(y - (C + DF)\hat{x} - Dr)$$

Uncontrollable/Unobservable Systems

Suppose the system to be stabilized is uncontrollable and/or unobservable.

Then the closed-loop system has an uncontrollable and/or unobservable part.

This part is invariant under state feedback and/or output injection.

As a result, this part is stable iff the system is both *stabilizable* and *detectable*.

The construction of F and K still holds under this assumption.

Comparison

Transfer function approach

- **find any doubly coprime matrix fractions for H_1**
- **solve the Bézout equations**
- **form H_2 using $W(s)$**

State space approach

- **construct a central stabilizing controller for S_1**
- **form S_2 using $W(q)$**

No need to construct the doubly coprime fractions, nor to solve the Bézout equations.

The central controller is given *directly* by F and K.

Example

Stabilize an integrator

$$S_1 : \ (0, 1, 1, 0), \qquad H_1(s) = 1/s$$

Transfer function approach:

$$H_1(s) = \frac{1}{s} = \left(\frac{1}{s+1}\right)\left(\frac{s}{s+1}\right)^{-1} = N_1 D_1^{-1} = \tilde{D}_1^{-1}\tilde{N}_1$$

$$XD_1 + YN_1 = \tilde{D}_1\tilde{X} + \tilde{N}_1\tilde{Y} = \frac{s}{s+1}X + \frac{1}{s+1}Y = 1$$

$$X = \tilde{X} = 1, \quad Y = \tilde{Y} = 1$$

$$H_2(s) = -\left(1 - \tfrac{s}{s+1}\overline{W}(s)\right)\left(1 + \tfrac{1}{s+1}\overline{W}(s)\right)^{-1}$$

For example, $\overline{W}(s) = 0$ **implies** $H_2(s) = -1$.

Example

Stabilize an integrator

$$S_1 : (0, 1, 1, 0), \qquad H_1(s) = 1/s$$

State space approach:

system $\quad \dot{x} = u,\ y = x$

observer $\quad \dot{\hat{x}} = u + K(y - \hat{x})$

feedback $\quad u = F\hat{x} + W(q)(y - \hat{x})$

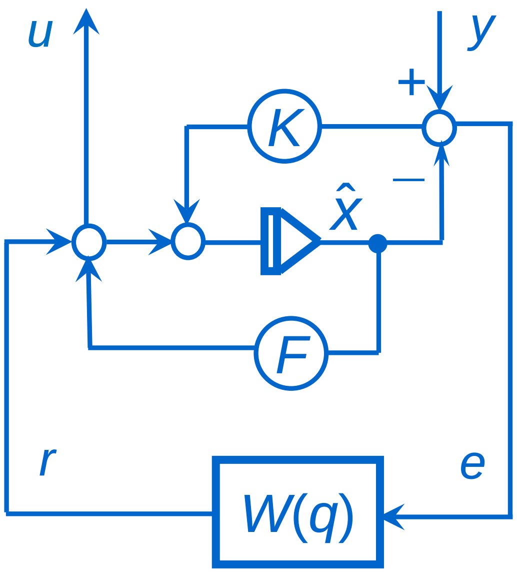
where F and K are any real numbers such that

$A + BF = F < 0$ and $A - KC = -K < 0$

Example

State space realization

of all controllers that stabilize an integrator



The set of stabilizing controllers contains controllers

of order *higher than any number* (= 1 + δW)

Example

The transfer function formula suggests there exist stabilizing controllers of order zero.

Indeed, *W* defined by

$$W(s) = -\frac{s+(K-F)+KF}{s+1}$$

yields the controller

$$H_2 := \left[\begin{array}{cc|c} F-K+1 & 1 & K-1 \\ -1+(K-F)+KF & -1 & 1-(K-F)-KF \\ \hline F+1 & 1 & -1 \end{array}\right] = -1$$

that is uncontrollable and unobservable, whose transfer function has McMillan degree zero. It corresponds to $\overline{W}(s) = 0$.

Example

To obtain the central stabilizing controller using the transfer function approach, put

$$H_1(s) = \tfrac{1}{s} = \left(\tfrac{s}{s+K}\right)^{-1}\left(\tfrac{1}{s+K}\right) = \left(\tfrac{s}{s-F}\right)\left(\tfrac{1}{s-F}\right)^{-1} = \tilde{D}_1^{-1}\tilde{N}_1 = N_1 D_1^{-1}$$

and solve for *strictly proper Y* and

$$X = \frac{s+(K-F)}{s+K},\ Y = \frac{-KF}{s+K}, \quad \tilde{X} = \frac{s+(K-F)}{s-F},\ \tilde{Y} = \frac{-KF}{s-F}$$

Then

$$H_2(s) = -\left(\frac{s+(K-F)}{s+K}\right)^{-1}\left(\frac{-KF}{s+K}\right) = -\left(\frac{-KF}{s-F}\right)\left(\frac{s+(K-F)}{s-F}\right)^{-1} = -X^{-1}Y = -\tilde{Y}\tilde{X}^{-1}$$

$$= \frac{KF}{s+(K-F)}$$

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OP Vzdělávání pro konkurenceschopnost

VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ

INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Optimal Control Systems with Prescribed Eigenvalues

Jiří Cigler

14. ledna 2010

Tato prezentace je spolufinancována Evropským sociálním fondem a státním rozpočtem České republiky.

Linear Quadratic Control

An optimal state-space design
given a performance index / weighting matrices.

The closed-loop transient behavior
difficult to determine in advance
as no simple relation exists
between the weighting matrices
and the closed-loop eigenvalues.

The weights to be determined iteratively
through trial and error.

Pole Placement Design

The closed-loop eigenvalues being specified,
the transient behavior can be addressed directly.

Not easy to transform transient requirements
into a set of closed-loop eigenvalues.

Different feedback gains
can lead to the same pole pattern
when the system has several inputs
and these gains can produce different transients.

A Remedy Proposed

Combine the LQ and pole placement designs as follows.

Start with a standard LQ design.

Should an undesirable closed-loop eigenvalue result,
select the weighting matrices so as to shift it
to a more convenient position
while leaving the remaining eigenvalues
at their original positions.

Repeat if desired.

Background

Attempts to modify the LQ design are of an early date.

The seminal work of Solheim (1972)
improved by Sugimoto and Yamamoto (1989)
and Kučera and Kraus (1999)
using different techniques.

The ultimate solution is reported by
Cigler (2009) and Cigler and Kučera (2009).

Preliminaries

Linear system $\dot{x}(t) = Ax(t) + Bu(t)$, n **states,** m **inputs.**

Performance index $\int_0^{\infty}(x^T Qx + u^T Ru)dt$, $Q = C^T C$, $R > 0$,

where (*A*, *B*) stabilizable and (*A*, *C*) detectable.

Optimal control law $u(t) = -R^{-1}B^T Px(t)$,

where $P \geq 0$ **is a unique matrix solution of the algebraic Riccati equation**

$$PA + A^T P - PBR^{-1}B^T P + Q = 0.$$

Hamiltonian

Consider the Hamiltonian

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}$$

The eigenvalues of *H* are symmetrically distributed with respect to the imaginary axis

and the *n* stable eigenvalues of *H* are the eigenvalues of the optimal closed-loop system.

Outline of the Presentation

Basic questions concerning

the LQ optimal eigenvalue shifting:

- **Which shifts are possible?**
- **Which weighting matrices *Q* and *R* realize the desired shift?**
- **How to solve the resulting Riccati equation for *P* ?**

We shall provide a complete answer.

Single Eigenvalue Relocation

Apply a similarity transformation *T* to bring *A*
to the Jordan form $\tilde{A}$
and choose one controllable eigenvalue, say λ_1, to be shifted

$$\tilde{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & J_1 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} b_1^T \\ \times \end{bmatrix}$$

Take

$$\tilde{Q} = \begin{bmatrix} q_1 & 0 \\ 0 & 0 \end{bmatrix}$$

with $q_1 \geq 0$ a real parameter,
and select the weighting matrix *R* so that $b_1^T R^{-1} b_1 = 1$.

Single Eigenvalue Relocation

Calculate

$$\det(sI - H) = \det(sI - \tilde{H})$$

$$= \det\left[\begin{array}{cc:cc} s-\lambda_1 & 0 & 1 & \times \\ 0 & sI - J_1 & \times & \times \\ \hdashline q_1 & 0 & s+\lambda_1 & 0 \\ 0 & 0 & 0 & s+J_1^T \end{array}\right]$$

$$= \det(sI - J_1)\det(sI + J_1^T)\det(sI - H_1)$$

where

$$H_1 = \begin{bmatrix} \lambda_1 & -1 \\ -q_1 & -\lambda_1 \end{bmatrix}$$

Single Eigenvalue Relocation

Let μ_1 be the desired position

to which the eigenvalue λ_1 is to be shifted.

Then

$$\det(sI - H_1) = s^2 - (\lambda_1^2 + q_1) = (s - \mu_1)(s + \mu_1)$$

We conclude that

$$|\mu_1| \geq |\lambda_1|$$

since $q_1 \geq 0$.

In particular, if μ_1 is to be stable,

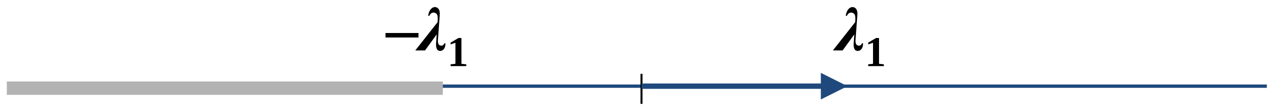
it can only be shifted to the left.

Single Eigenvalue Relocation

The eigenvalue can only be shifted to the left

Note that when λ_1 is not stable

it is shifted to the left of its stable image $-\lambda_1$



Single Eigenvalue Relocation

Having chosen *Q* and *R*, the optimal control law that achieves the desired shift is given by solving the Riccati equation.

Make an inspired guess that

$$\tilde{P} = \begin{bmatrix} p_1 & 0 \\ 0 & 0 \end{bmatrix}$$

for some real constant $p_1 \geq 0$.

The Riccati equation then reduces to

$$p_1^2 - 2\lambda_1 p_1 - q_1 = 0$$

which can readily be solved. In particular, $p_1 = \lambda_1 - \mu_1$.

Single Eigenvalue Relocation

In the original coordinates,

the solution of the Riccati equation is

$$P = (T^{-1})^T \tilde{P} T^{-1}$$

and the optimal control law that achieves the desired shift is given by

$$u(t) = -R^{-1} B^T P x(t).$$

Multiple Eigenvalue Relocation

Suppose that λ_1 is not simple
but generates a Jordan block of size k.

The previous result holds also in this case.
While λ_1 is shifted to μ_1, λ_1 remains an eigenvalue of A
but it generates a Jordan block of size $k - 1$.

The process can be continued
to result in a spectrum of k eigenvalues $\mu_1, \mu_2, \ldots, \mu_k$
positioned to the left of the value $-|\lambda_1|$.

A Pair of Complex Conjugate Eigenvalues

Suppose a complex conjugate pair of controllable eigenvalues $\lambda_1 = \lambda$ and $\lambda_2 = \bar{\lambda}$ is to be shifted to obtain a new pair $\mu_1 = \mu$ and $\mu_2 = \bar{\mu}$.

Following a similarity transformation, one obtains

$$\tilde{A} = \begin{bmatrix} \Lambda_2 & 0 \\ 0 & J_2 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_2^T \\ \times \end{bmatrix}$$

where

$$\Lambda_2 = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}$$

A Pair of Complex Conjugate Eigenvalues

Take

$$\tilde{Q} = \begin{bmatrix} Q_2 & 0 \\ 0 & 0 \end{bmatrix}$$

where

$$Q_2 = \begin{bmatrix} q & q_{12} \\ \overline{q}_{12} & q \end{bmatrix}$$

for a real q and a complex q_{12} that satisfy $q \geq |q_{12}|$.

Select R so that

$$B_2^T R^{-1} \overline{B}_2 = \begin{bmatrix} 1 & \overline{\omega} \\ \omega & 1 \end{bmatrix} := \Omega_2$$

for a complex ω such that $|\omega| \leq 1$.

A Pair of Complex Conjugate Eigenvalues

The relevant 4×4 Hamiltonian matrix

$$H_2 = \begin{bmatrix} \Lambda_2 & -\Omega_2 \\ -Q_2 & -\overline{\Lambda}_2^T \end{bmatrix}$$

is to have eigenvalues $\mu, \bar{\mu}$ and $-\mu, -\bar{\mu}$.

Comparing the determinants,

the region in which $\mu, \bar{\mu}$ can be placed

is given by

$$\mathrm{Re}\,\mu^2 = \mathrm{Re}\,\lambda^2 + q + \mathrm{Re}\,\omega q_{12}$$

$$|\mu|^4 = |\lambda|^4 + 2|\lambda|^2 q + 2\,\mathrm{Re}\,\lambda^2 \omega q_{12} + (1 - |\omega|^2)(q^2 - |q_{12}|^2).$$

A Pair of Complex Conjugate Eigenvalues

The complex case is more complicated.

The allowable region for $\mu, \bar{\mu}$ depends, for a given pair $\lambda, \bar{\lambda}$, upon ω.

We shall distinguish three cases:

- $|\omega| = 1$
- $\omega = 0$
- $0 < |\omega| < 1$

A Pair of Complex Conjugate Eigenvalues

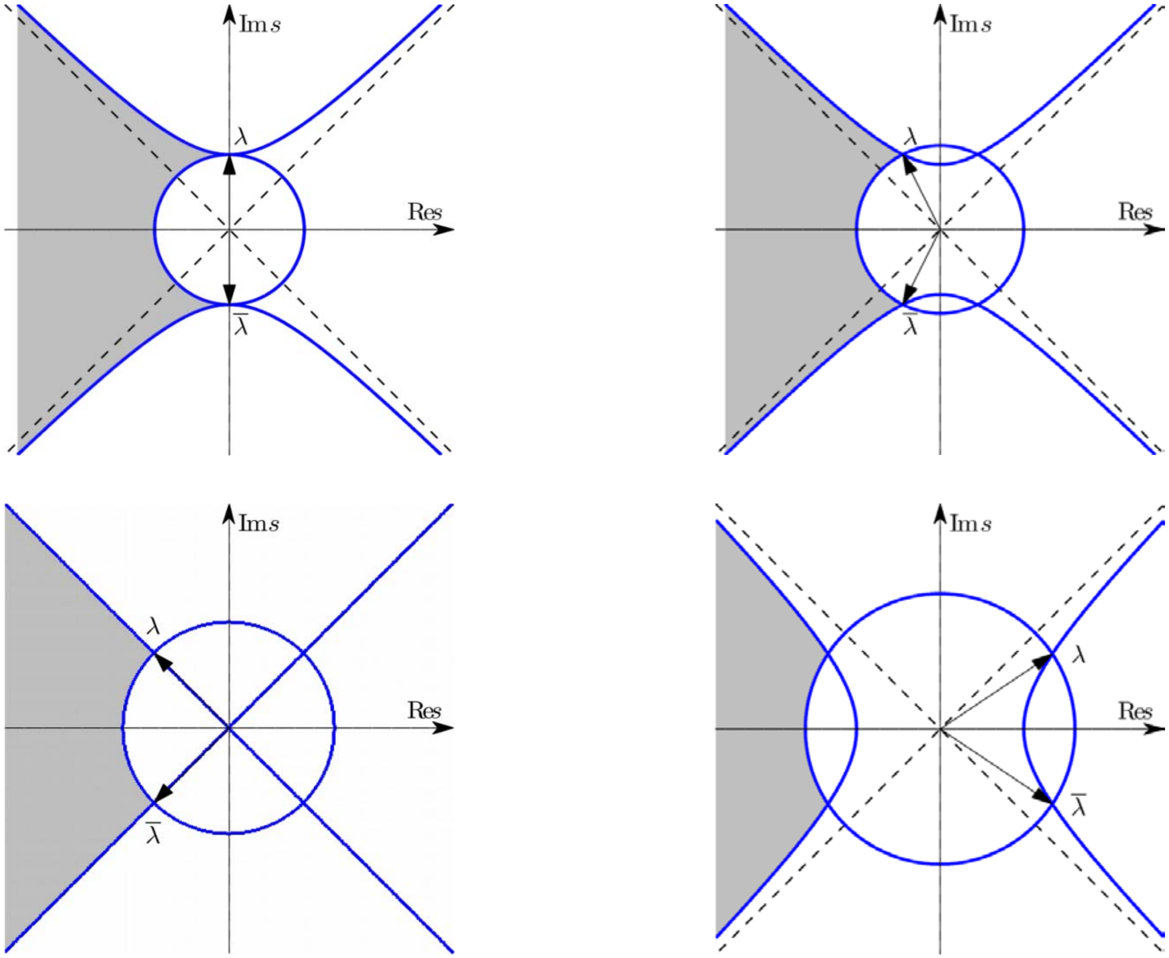
When $|\omega| = 1$, then Ω_2 is a singular rank-one matrix and the equations can be simplified to

$$x^2 - y^2 \geq \mathrm{Re}\,\lambda^2$$
$$x^2 + y^2 \geq |\lambda|^2$$

The first equation describes the interior/exterior of a hyperbola, depending on the sign of $\mathrm{Re}\,\lambda^2$

while the second one represents the exterior of a circle with radius $|\lambda|$, centered at the origin.

A Pair of Complex Conjugate Eigenvalues



A Pair of Complex Conjugate Eigenvalues

When $\omega = 0$, then Ω_2 is the identity matrix and the equations can be simplified to

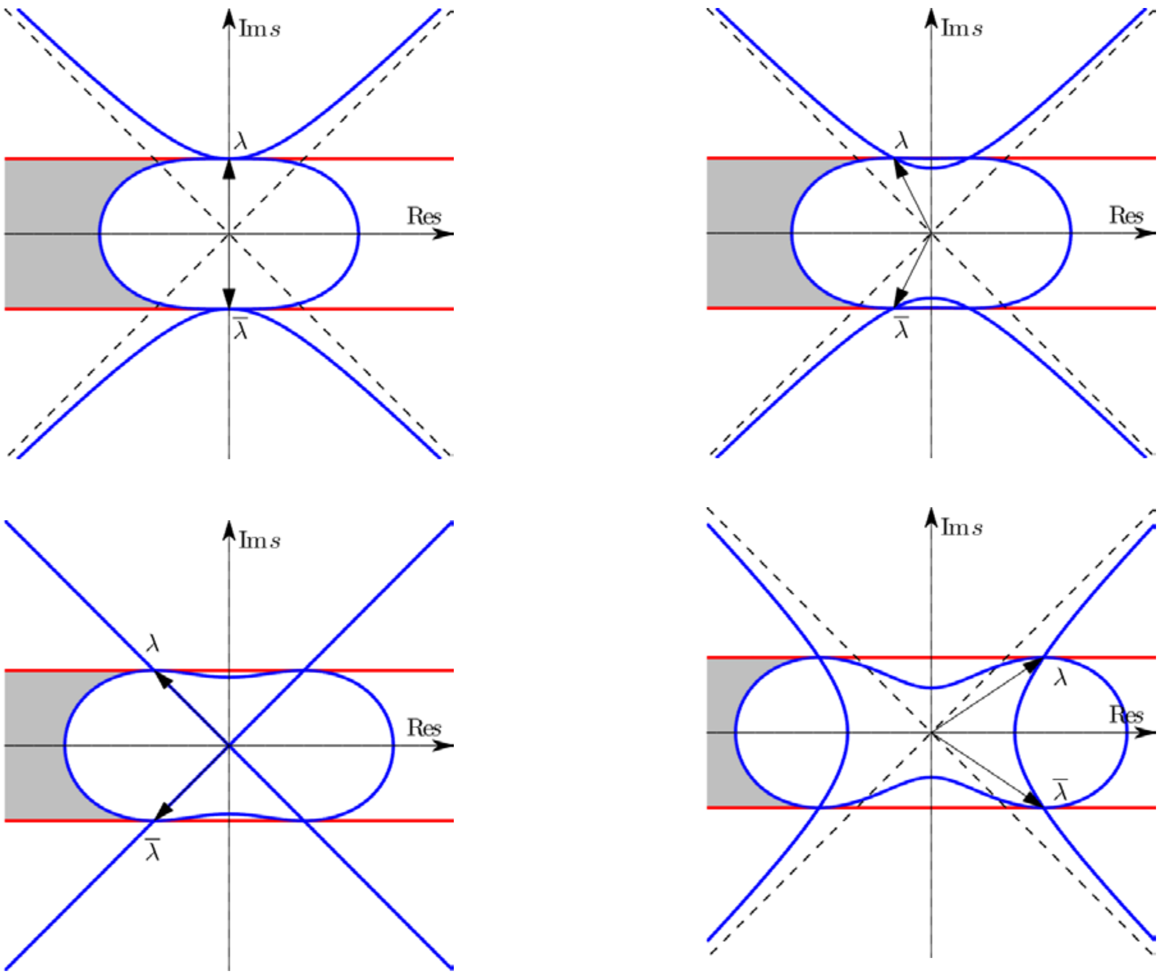
$$y^2 \le \mathrm{Im}^2 \lambda$$

$$(x^2 + y^2)^2 - 2|\lambda|^2 (x^2 - y^2) + |\lambda|^4 \ge 4|\lambda|^2 \mathrm{Im}^2 \lambda$$

The first equation defines a strip along the real axis

while the second one represents the exterior of a Cassini oval with foci at λ and –λ, whose shape depends on Im λ / |λ|.

A Pair of Complex Conjugate Eigenvalues



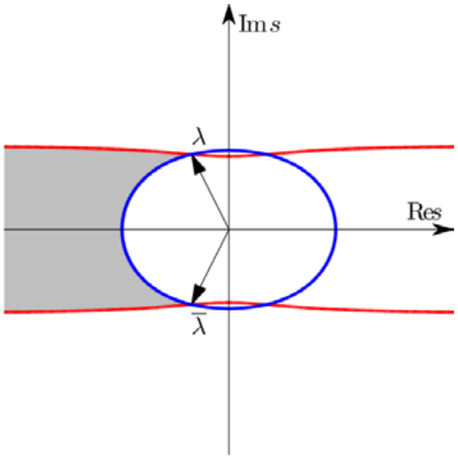
A Pair of Complex Conjugate Eigenvalues

When $0 < |\omega| < 1$, then Ω_2 is a general rank-two matrix and the regions the equations define can most conveniently be found using optimization techniques to obtain upper bounds for the real and imaginary parts of μ.

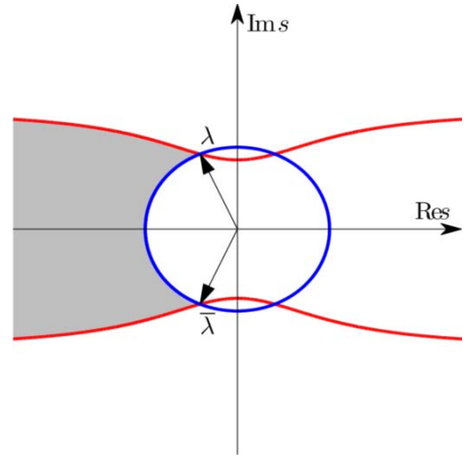
When $\omega \to 0$ and/or $|\omega| \to 1$, the regions thus defined approach those considered previously.

A Pair of Complex Conjugate Eigenvalues

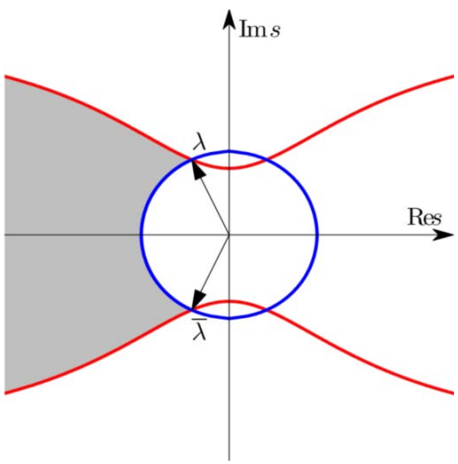
$|\omega| = 0.5$



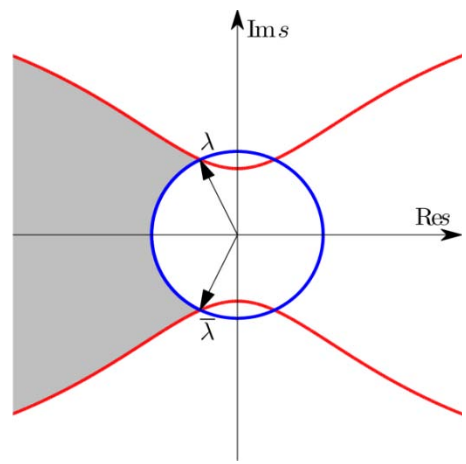
$|\omega| = 0.8$



$|\omega| = 0.9$



$|\omega| = 0.95$



A Pair of Complex Conjugate Eigenvalues

Having chosen Q and R, the optimal control law that achieves the desired shift is given by solving the Riccati equation.

Make an inspired guess that

$$\tilde{P} = \begin{bmatrix} P_2 & 0 \\ 0 & 0 \end{bmatrix}$$

where $P_2 \geq 0$ is a Hermitian 2×2 matrix that solves the reduced Riccati equation

$$P_2 \Lambda_2 + \overline{\Lambda}_2^T P_2 - P_2 \Omega_2 P_2 + Q_2 = 0$$

A Pair of Complex Conjugate Eigenvalues

In the original coordinates,

the solution of the Riccati equation is

$$P = (T^{-1})^T \tilde{P} T^{-1}$$

and the optimal control law that achieves the desired shift is given by

$$u(t) = -R^{-1} B^T P x(t).$$

Example: Oscillatory System (1)

Consider the following oscillatory system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

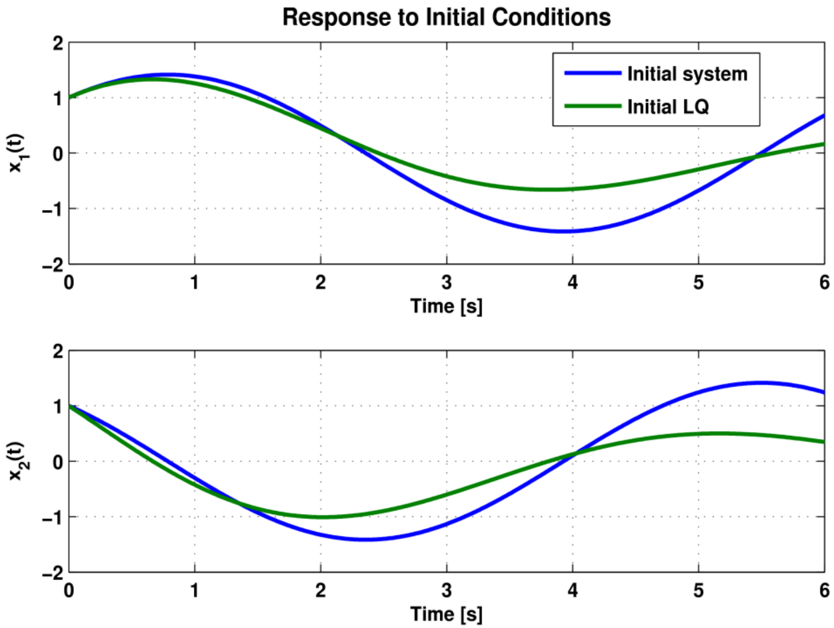
```
%System matrices
>>  A = [0 1; -1 0];
>> B = [0 1]';
>> C = [1 0; 0 1] ;
>> D = [0 0]';
```

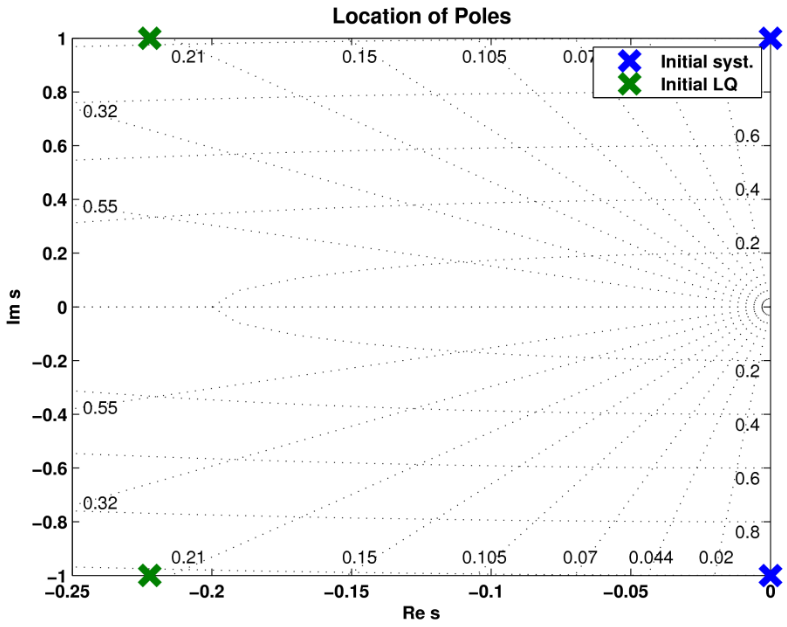
Initial guess of weighting matrices

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R = 10$$

gives the complex conjugate poles at $\lambda = -0.22 \pm j0.99$

Example: Oscillatory System (2)



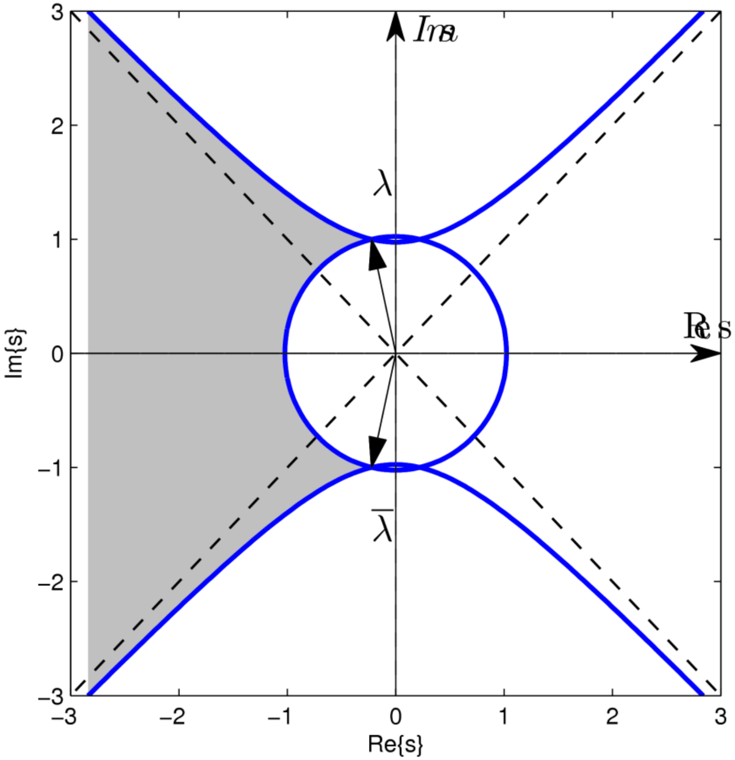



```
>>Q0 = eye(2);
>>R0 = 10;
>>P0 = care(A,B,Q0,R0);                    % solve ARE
>>F0 = -inv(R0)*B'*P0;                     % compute feedback gain
>>SYS1 = ss(A+B*F0,B, C+D*F0,D);  % feedback system
>>[A1,B1,C1,D1]=ssdata(SYS1);       % get system matrices
```

Example: Oscillatory System (3)

The allowable area is defined by

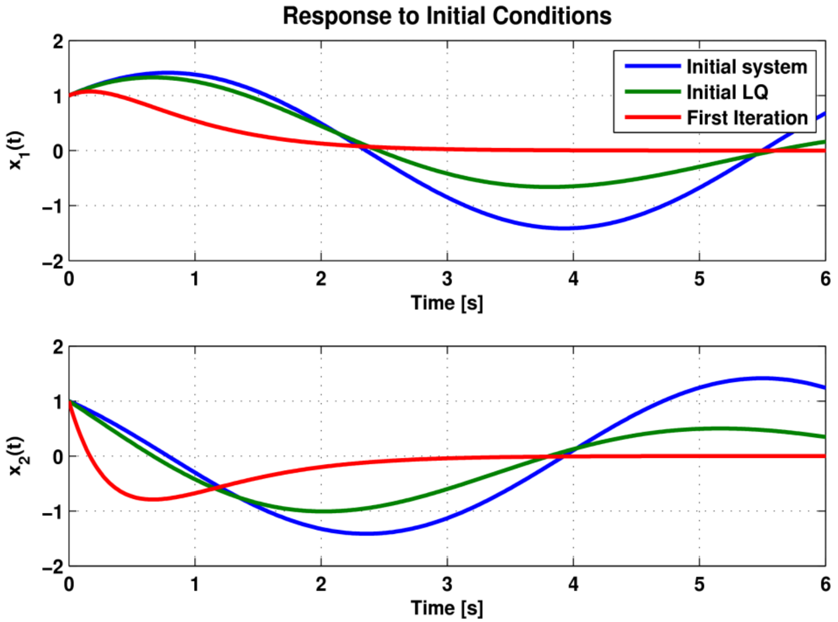
- **The interior of the hyperbola**
- **The exterior of the circle with radius $|\lambda|$, centered at the origin**

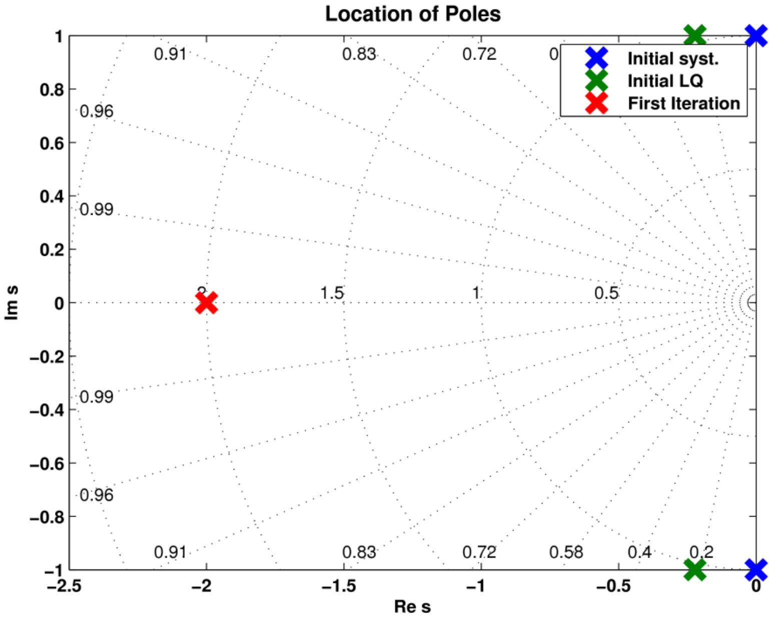


Our objective:

- **Shift poles to increase damping**
- **New location $\lambda_{1,2} = -2$**

Example: Oscillatory System (4)






```
>>[T,J]=jordan(A1);                    % Compute Jordan form
>>W = inv(T);
>> B1t = W*B1;                         % Transformed B
>> B1prod = B1t*B1t';
>> R1 = B1prod(1);                     % Compute R1
>> omega = inv(R1) * B1prod(2,1);
>> Q1 = inv(T)'*solve(%%equations for q, q12%% )*inv(T);
```

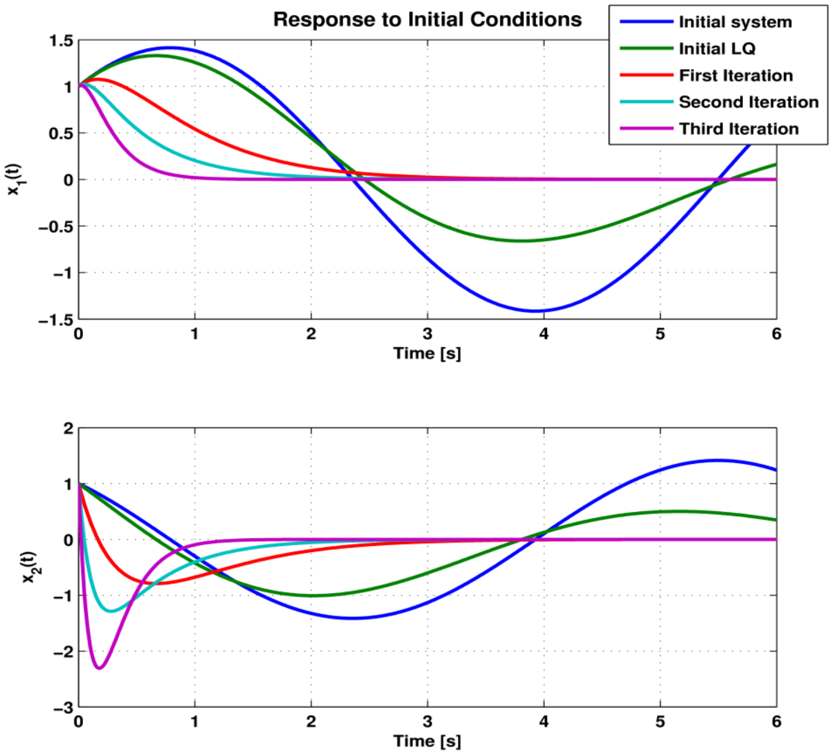
Example: Oscillatory System (5)

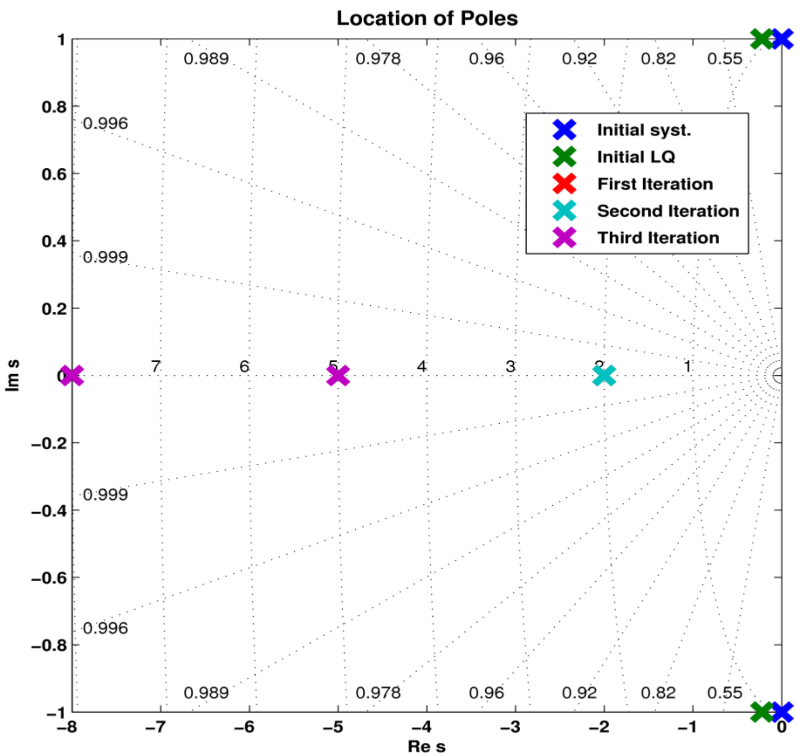
Poles have been shifted to the new location.

Now, make the system faster.

- Shift one pole to $\lambda_1 = -5$
- The second one to $\lambda_2 = -8$

Example: Oscillatory System (6)






```
>> q = mu^2 - lambda1^2;
>> p = lambda-mu;
>> P2 = W(1,:)'*p*W(1,:);
>> Q2 = W(1,:)'*q*W(1,:);
```

Conclusions

A method has been proposed
that makes use of the LQ optimization
to shift a single eigenvalue
or a pair of complex conjugate eigenvalues
in an iterative manner.

The region into which each eigenvalue can be shifted
has been described in detail.

A simple and transparent method
that will eventually make its way to textbooks.

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Is Deadbeat Control l_2 Optimal?

Vladimír Kučera

14. ledna 2010

Tato prezentace je spolufinancována Evropským sociálním fondem a státním rozpočtem České republiky.

Introduction

A typical linear control strategy
in discrete-time systems,
***deadbeat control* produces transients**
that vanish in finite time.

On the other hand,
the *linear-quadratic control* stabilizes the system
and minimizes the l_2 norm of its transient response.

Quite surprisingly,
it is shown that *deadbeat systems are l_2 optimal*,
at least for reachable systems.

Deadbeat Control

Given a linear system (*A*, *B*)

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \ldots$$

where $u_k \in \mathcal{R}^m$ **and** $x_k \in \mathcal{R}^n$.

The objective of *deadbeat regulation* is to determine a linear state feedback of the form

$$u_k = -Lx_k$$

that drives each initial state x_0 **to the origin in a least number of steps.**

Reachability and Controllability

We define the *reachability subspaces* by

$$R_0 = 0,$$
$$R_k = \text{range}[B \;\; AB \;\; \ldots \;\; A^{k-1}B], \quad k = 1, 2, \ldots .$$

When $R_n = \mathcal{R}^n$, the system (A, B) is said to be *reachable*.

The system (A, B) is said to be *controllable* if there exists a basis in which

$$A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

where (A_1, B_1) is reachable and A_2 is nilpotent.

Reachability Indices

For each $k = 1, 2, ...$

let $S_1, S_2, ..., S_k$ be a sequence of matrices such that

$$\text{range}[BS_1 \;\; ABS_2 \;\; \dots \;\; A^{k-1}BS_k] = R_k$$

Therefore $S_1, S_2, ..., S_k$ serve to *select a basis* for R_k.

The *reachability indices* $r_1, r_2, ..., r_m$ are defined by

$$r_i = \text{cardinality}\{S_j,\; j = 1,2,\dots : \text{rank } S_j \geq i\}$$

Note that r_i is the number of times

dimension increases by at least i in the sequence $R_0, R_1, R_2, ...$

Theorem 1

There exists a deadbeat control law

if and only if the system (A, B) is controllable.

Let

$$L_0 = 0,$$
$$L_k = L_{k-1} + L'_k (A - BL_{k-1})^k, \quad k = 1, 2, \ldots$$

where L'_k satisfies

$$L'_k [BS_1 \ \ ABS_2 \ \ \ldots \ \ A^{k-1}BS_k] = [0 \ \ \ldots \ \ 0 \ \ S_k].$$

Then $L = L_n$ is a deadbeat regulator gain.

The closed-loop system matrix $A - BL$ is nilpotent.

Example

Consider a system given by

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and determine all deadbeat regulator gains.

The reachability subspaces are

$$R_0 = 0, \; R_1 = \mathrm{range}\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \; R_2 = \mathrm{range}\left[\begin{array}{cc:cc} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right], \ldots$$

A basis for R_k is selected using $S_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; S_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Example

Recursive procedure to determine all deadbeat gains

$$L_0 = 0$$

$$L_1 = L_0 + L_1'(A - BL_0), \qquad L_1' BS_1 = S_1$$

$$= \begin{bmatrix} \alpha & 0 & \alpha - 1 \\ \beta & 0 & \beta + 1 \end{bmatrix} \qquad L_1' = \begin{bmatrix} \alpha & 0 & -1 \\ \beta & 0 & 1 \end{bmatrix}$$

$$L_2 = L_1 + L_2'(A - BL_1)^2, \qquad L_2'\begin{bmatrix} BS_1 & ABS_2 \end{bmatrix} = \begin{bmatrix} 0 & S_2 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha & 0 & \alpha - 1 \\ 1 & 0 & 2 \end{bmatrix} \qquad L_2' = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$L_3 = L_2 = L$$

Example

The feedback system matrix is nilpotent,

$$A - BL = \begin{bmatrix} 1 & 0 & 1 \\ -1-\alpha & 0 & -1-\alpha \\ -1 & 0 & -1 \end{bmatrix}, \quad (A-BL)^2 = 0$$

and any initial state is transferred to zero in three steps,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} x+z \\ -(1+\alpha)(x+z) \\ -(x+z) \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Note that deadbeat gains are not unique, α any real number.

Linear Quadratic Regulator

Given a linear system (A, B)

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \ldots$$

where $u_k \in \mathcal{R}^m$ **and** $x_k \in \mathcal{R}^n$.

The objective of LQ regulation

is to find a linear state feedback of the form

$$u_k = -Lx_k$$

that stabilizes the closed-loop system

and, for every initial state x_0, minimizes the l_2 norm

of a specified output $y_k \in \mathcal{R}^m$ **of the form**

$$y_k = Cx_k + Du_k$$

Equivalent Formulation

Note that the l_2 norm of the output

$$y_k = Cx_k + Du_k$$

is given by

$$\|y\|_2^2 = \sum_{k=0}^{\infty} y_k^T y_k = \sum_{k=0}^{\infty} (x^T W x + 2x^T V u + u^T U u),$$

where

$$W = C^T C,\ V = C^T D,\ U = D^T D$$

are weighting matrices of the quadratic performance index to be minimized.

Stabilizability and Invertibility

The system (A, B) is said to be *stabilizable*

if there exists a basis in which

$$A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

where (A_1, B_1) is reachable and A_2 is stable.

The system (A, B, C, D)

$$x_{k+1} = Ax_k + Bu_k,\ y_k = Cx_k + Du_k, \quad k = 0, 1, \ldots$$

is said to be *left invertible*

if its transfer function has full column normal rank.

Invariant Zeros

We further define the *system matrix* as the polynomial matrix

$$S(z) = \begin{bmatrix} zI_n - A & -B \\ C & D \end{bmatrix}$$

and say that a complex number ζ is an *invariant zero* of the system (*A*, *B*, *C*, *D*) if the rank of *S*(ζ) is less than the normal rank of *S*(*z*).

Theorem 2

Suppose that the system (*A*, *B*) is stabilizable.
Suppose that the system (*A*, *B*, *C*, *D*) is left invertible and also has no invariant zeros on the unit circle $|z| = 1$.

Then, there exists a *unique* LQ regulator gain given by

$$L = (D^T D + B^T XB)^{-1}(B^T XA + D^T C),$$

where *X* is the largest non-negative definite solution of the algebraic Riccati equation

$$X = A^T XA + C^T C - (B^T XA + D^T C)^T (D^T D + B^T XB)^{-1}(B^T XA + D^T C)$$

Reachability Standard Form

Let system (A, B) be reachable,

with reachability indices $r_1, r_2, \ldots, r_m$.

Then there exists a similarity transformation T

that brings the matrices A and B

to the *reachability standard form*,

$$A' = TAT^{-1}, \quad B' = TB$$

where A' is a block diagonal of $r_i \times r_i$ top-companion matrices

with nonzero entries in rows r_i, $i = 1, 2, \ldots, m$

and B' has nonzero entries

only in rows r_i and columns $j \geq i$, $i = 1, 2, \ldots, m$.

Reachability Standard Form

For example,

when $n = 5$ and $m = 2$, with reachability indices $r_1 = 3$, $r_2 = 2$,

the reachability standard form looks like

$$A' = \left[\begin{array}{ccc:cc} & 1 & & & \\ & & 1 & & \\ \times & \times & \times & \times & \times \\ \hdashline & & & & 1 \\ \times & \times & \times & \times & \times \end{array}\right], \quad B' = \left[\begin{array}{cc} & \\ & \\ 1 & \times \\ \hdashline & \\ & 1 \end{array}\right]$$

where the empty positions are zeros

and × indicates specific coefficients.

Theorem 3

Suppose that the system (*A*, *B*) is reachable,
with reachability indices $r_1 \geq r_2 \geq \ldots \geq r_m$
and with the matrix *B* having rank *m*.

Let *T* be a similarity transformation
that brings *A* and *B* to the reachability standard form.

Then, the feedback gain *L*
that is LQ optimal with respect to $C = T$ and $D = 0$
is a deadbeat gain.

Proof: Existence

We first show that an LQ regulator gain exists that is optimal with respect to $C = T$ and $D = 0$.

Indeed, the system (A, B) is reachable hence stabilizable.

The system (A, B, T, 0) has a transfer function whose normal rank is m, so it is left invertible.

The system matrix

$$S(z) = \begin{bmatrix} zI_n - A & -B \\ T & 0 \end{bmatrix}$$

has rank $n + m$ for all complex numbers z, hence (A, B, T, 0) has no invariant zeros at all.

The assumptions of Theorem 2 *are all satisfied*.

Proof: Polynomial Matrix Fractions

Write the transfer function of the system (A, B) in the polynomial matrix fraction form

$$(zI_n - A)^{-1}B = Q(z)P^{-1}(z)$$

For any feedback applied to the system, one obtains

$$[zI_n - (A - BL)]^{-1}B = Q(z)[P(z) + LQ(z)]^{-1}$$

The system (A, B) being reachable,

these polynomial matrix fractions are coprime.

Thus the matrices $zI_n - (A - BL)$ and $P(z) + LQ(z)$ have the *same invariant factors*.

Proof: Matrix Identity

Consider the algebraic Riccati equation

and introduce the optimal feedback gain L

for $(D^T D + B^T XB)^{-1}(B^T XA + D^T C)$.

Add $z^{-1}(PA - PA) + z(A^T P - A^T P)$ **to both sides,**

multiply by $(zI - A)^{-1}B$ on the right

and by $B^T(z^{-1}I - A^T)^{-1}$ on the left,

and use the matrix fractions

$$(zI_n - A)^{-1}B = Q(z)P^{-1}(z)$$

to introduce the polynomial matrices P and Q.

Proof: Matrix Identity

Complete the squares to obtain the matrix identity

$$[P(z^{-1})+LQ(z^{-1})]^T(D^TD+B^TXB)[P(z)+LQ(z)]$$
$$=[CQ(z^{-1})+DP(z^{-1})]^T[CQ(z)+DP(z)]$$

Define a polynomial matrix F by the equation

$$F^T(z^{-1})F(z)=[CQ(z^{-1})+DP(z^{-1})]^T[CQ(z)+DP(z)]$$

in such a way that F^{-1} is analytic in the domain $|z|\geq 1$.

Such a matrix F is referred to as the *spectral factor.*

Proof: Reachability Standard Form

Bring (*A*, *B*) to the *reachability standard form* using the similarity transformation matrix *T*. The corresponding polynomial fraction matrices are related by

$$P'(z) = P(z), \quad Q'(z) = TQ(z)$$

and Q' has the block-diagonal form

$$Q'(z) = \text{block-diag}\left[\begin{bmatrix} 1 \\ z \\ \vdots \\ z^{r_1-1} \end{bmatrix}, \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{r_2-1} \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{r_m-1} \end{bmatrix} \right].$$

Proof: Spectral Factorization

The spectral factorization reads

$$F^T(z^{\ 1})F(z) = Q^T(z^{\ 1})T^T TQ(z)$$

$$= Q'^T(z^{\ 1})Q'(z) = \mathrm{diag}\,[r_1, r_2, \ldots, r_m]$$

so that

$$F(z) = \mathrm{diag}\,[\sqrt{r_1}z^{r_1}, \sqrt{r_2}z^{r_2}, \ldots, \sqrt{r_m}z^{r_m}].$$

The matrices $P(z) + LQ(z)$ and $zI_n - (A - BL)$ share the same invariant factors $z^{r_1}, z^{r_2}, \ldots, z^{r_m}$.

Therefore, $A - BL$ is *nilpotent* with Jordan structure comprising m nilpotent blocks of sizes $r_1, r_2, \ldots, r_m$. This proves that L is a deadbeat gain.

Feedback Gain Calculation

Determine the similarity matrix *T* as follows

$$\left[BS_1 \ \ ABS_2\right]^{-1} = \left[\begin{array}{ccc} 1 & 1 & -1 \\ \hline -1 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right] = \begin{bmatrix} q_1 \\ \times \\ q_2 \end{bmatrix} \qquad \begin{array}{l} \leftarrow \text{row } r_1 \\ \\ \leftarrow \text{row } r_1 + r_2 \end{array}$$

$$T = \begin{bmatrix} q_1 \\ q_2 \\ q_2 A \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Then

$$A' = TAT^{-1} = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -1 & 2 \end{array}\right], \quad B' = TB = \left[\begin{array}{cc} 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right]$$

Feedback Gain Calculation

Determine a right polynomial matrix fraction

$$(zI - A')^{-1}B' = Q'(z)P'^{-1}(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & z \end{bmatrix} \begin{bmatrix} z & -1 \\ 0 & z^2 - 2z + 1 \end{bmatrix}^{-1}$$

and a spectral factor

$$F(z) = \begin{bmatrix} z & 0 \\ 0 & \sqrt{2}z \end{bmatrix}$$

Solve the matrix polynomial equation $XP'(z) + YQ'(z) = F(z)$

and put $L' = X^{-1}Y$; **then**

$$L = L'T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

which corresponds to $\alpha = 1$.

Conclusions

- Deadbeat control and LQ regulation,
  two strategies different in nature, are in fact related.
- A deadbeat control law can be obtained
  by solving a particular LQ regulator problem,
  at least for reachable systems.
- The LQ optimal regulator gain is unique,
  whereas the deadbeat feedback gains are not.
  Only one deadbeat gain is LQ optimal.
- An alternative construction of such a gain
  is thus available, solving the Riccati equation.

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